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INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTRO--ETC(U)
MAY 79 R E BECHHOFER , A C TAMHANE DAAG29-77-C-0003

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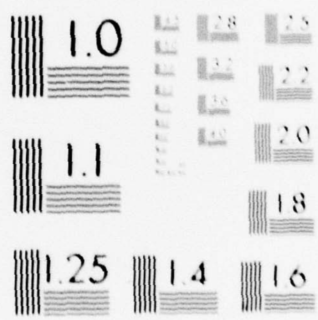

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9 TECHNICAL REPORT NO. 425

11 May 1979

12 73

6 INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL (II): II.
OPTIMAL DESIGNS FOR $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$.

by

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14 TR-425

Research supported by 15

U.S. Army Research Office-Durham contract DAAG29-77-~~C~~-0003,

Office of Naval Research Contract N00014-75-C-0586

at Cornell University

and

NSF Grant ENG 77-06112

at Northwestern University

409 869

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ABSTRACT

In this paper we continue the study of balanced treatment incomplete block (BTIB) designs, ~~which we initiated in [3]~~. These designs are appropriate for comparing simultaneously $p \geq 2$ > or = test treatments with a control treatment--the so-called multiple comparisons with a control (MCC) problem. ~~This class of designs was characterized in [3].~~ In the present paper we obtain optimal designs within this class for selected (p,k,b) where $k < p+1$ is the number of plots per block, and b is the total number of blocks available. Specifically, optimal designs are obtained for $p = 2(1)6$, $k = 2$ and for $p = 3$, $k = 3$.

Tables of exact (discrete) optimal designs are given for these (p,k)-values for a range of b-values which would ordinarily be of practical interest. Tables of approximate (continuous) optimal designs are given for situations in which very large b-values are required. The theory underlying these approximate designs is developed, and the goodness of the approximation is studied.

Key words and phrases: Multiple comparisons with a control, balanced treatment incomplete block (BTIB) designs, admissible designs, optimal designs, applications of equicorrelated multivariate normal and multivariate Student's t distributions.

1. INTRODUCTION AND SUMMARY

In our first paper [3] we initiated the study of balanced treatment incomplete block (BTIB) designs which are appropriate for comparing simultaneously $p \geq 2$ test treatments with a control treatment. This class of designs was characterized in [3]; in the present paper we obtain optimal designs within this class for selected (p,k,b) where $k < p+1$ is the number of plots per block, and b is the total number of blocks available for experimentation.

In order to make this paper self-contained we will state the key definitions and results from [3]. We shall use the following notation (also used in [3]): Let the treatments be indexed by $0,1,\ldots,p$ with 0 denoting the control treatment and $1,2,\ldots,p$ denoting the test treatments. The $N = kb$ experimental units can be arranged in b blocks each of size k. If treatment i is assigned to the hth plot of the jth block $(0 \leq i \leq p,\ 1 \leq h \leq k;\ 1 \leq j \leq b)$, let Y_{ijh} denote the corresponding random variable; we assume the usual additive linear model (no treatment×block interaction)

$$Y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} \tag{1.1}$$

with $\sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$; the e_{ijh} are assumed to be i.i.d. $N(0,\sigma^2)$ random variables. It is desired to make an exact joint confidence statement (employing one-sided or two-sided intervals) concerning the p differences $\alpha_0 - \alpha_i$ based on their best linear unbiased estimators (BLUE's) $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

In Section 3.1 of [3] we proposed a class of incomplete block designs which are balanced with respect to the test treatments in the following sense: $\mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \eta^2\sigma^2/N$ $(1 \leq i \leq p)$ and $\mathrm{Corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \rho$ $(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p)$; the parameters η and ρ depend on the design employed. We refer to designs with this property as BTIB designs. Conditions that a design must satisfy in order that it be BTIB were given in Theorem 3.1 of [3]. This theorem states that if $\{r_{ij}\}$ is the incidence matrix of the design, r_{ij} being the total number of times the ith treatment appears in the jth block, and if $\lambda_{i_1 i_2} = \sum_{j=1}^{b} r_{i_1 j} r_{i_2 j}$ is the total number of times that the i_1th treatment appears with the i_2th treatment in the same block over the whole design $(i_1 \neq i_2;\ 0 \leq i_1, i_2 \leq p)$, then the necessary and sufficient conditions for a design to be BTIB are that

$$\lambda_{01} = \lambda_{02} = \ldots = \lambda_{0p} = \lambda_0 \quad \text{(say)}$$

$$\lambda_{12} = \lambda_{13} = \ldots = \lambda_{p-1,p} = \lambda_1 \quad \text{(say)}.$$

In Section 4 of [3] we restricted consideration to BTIB designs, and showed how to use such designs for experiments leading to joint one-sided (or two-sided) confidence interval estimates of the $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$ when σ^2 is known or unknown.

The concept of an admissible design (see Section 3.1 below) for such experiments was introduced in Section 5 of [3]. The problem of finding an optimal design in the class of admissible BTIB designs when (p,k,b) are given was mentioned in Section 4 of [3] but was not precisely formulated. We do so in Section 3.2 of the present paper, and solve the problem

for the cases $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$; additional (p,k)-combinations of practical interest with $k \geq 3$ will be considered in [4].

The specific MCC problem with which we are concerned in the present paper is that of obtaining joint <u>one-sided</u> confidence intervals of the form

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d \quad (1 \leq i \leq p) \tag{1.2}$$

for given values of (p,k,b) when σ^2 is <u>known</u>, and $d > 0$ is a <u>specified</u> "yardstick" associated with the "width" of the confidence interval. For this problem we seek an <u>optimal</u> design in the class of all admissible BTIB designs, an optimal design being one which <u>maximizes</u> the confidence coefficient associated with (1.2).

2. AN INTEGRAL EXPRESSION FOR THE CONFIDENCE COEFFICIENT

For ease of reference we record here the expressions derived in [3] for the estimators $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$, and their variances and correlations. Let T_i denote the sum of all observations obtained with the ith treatment $(0 \leq i \leq p)$, and let B_j denote the sum of all observations in the jth block $(1 \leq j \leq b)$. Define $B_i^* = \sum_{j=1}^{b} r_{ij}B_j$ and let $Q_i = kT_i - B_i^*$ $(0 \leq i \leq p)$. Then

$$\hat{\alpha}_0 - \hat{\alpha}_i = \frac{\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0(\lambda_0 + p\lambda_1)} \quad (1 \leq i \leq p). \tag{2.1}$$

Also,

$$\mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \frac{\eta^2}{N}\sigma^2 \quad (1 \leq i \leq p) \tag{2.2}$$

where

$$\eta^2 = \frac{k^2 b(\lambda_0+\lambda_1)}{\lambda_0(\lambda_0+p\lambda_1)} , \qquad (2.3)$$

and

$$\rho = \mathrm{Corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \frac{\lambda_1}{\lambda_0 + \lambda_1} \quad (i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p). \qquad (2.4)$$

The probability associated with (1.2) is given by

$$P\{\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d \quad (1 \leq i \leq p)\}$$

$$= P\{Z_i \leq d\sqrt{N}/\sigma\eta \quad (1 \leq i \leq p)\} \qquad (2.5)$$

where

$$Z_i = \frac{(\hat{\alpha}_0-\hat{\alpha}_i) - (\alpha_0-\alpha_i)}{\sigma\eta/\sqrt{N}} \quad (1 \leq i \leq p).$$

The Z_i have a standard p-variate normal distribution with $\mathrm{Corr}\{Z_{i_1}, Z_{i_2}\} = \rho$ $(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p)$; η^2 and ρ are given by (2.3) and (2.4), respectively. We next define

$$\xi = d\sqrt{N}/\sigma \qquad (2.6)$$

which is a pure number involving the <u>specified</u> quantity d. Then (2.5) can be written as

$$P\{Z_i \leq \xi/\eta \quad (1 \leq i \leq p)\} = \int_{-\infty}^{\infty} \Phi^p\left(\frac{x\sqrt{\rho} + \xi/\eta}{\sqrt{1-\rho}}\right) d\Phi(x) \qquad (2.7)$$

where $\Phi(\cdot)$ represents the standard normal cdf.

3. FORMULATION OF THE EXACT (DISCRETE) OPTIMIZATION PROBLEM

3.1 Inadmissible, admissible, equivalent and generator designs

Preliminary to posing our optimization problem we recall some definitions from [3]. If for given (p,k,b) we have two BTIB designs D_1 and D_2 with (η_1^2,ρ_1) and (η_2^2,ρ_2), respectively, and if $\eta_1^2 \leq \eta_2^2$ and $\rho_1 \geq \rho_2$ with at least one inequality being strict, then D_2 is said to be inadmissible since it is dominated (in the sense of having a smaller confidence coefficient for every d and σ) by D_1; if a design is not inadmissible, then it is said to be admissible. (A more general definition of inadmissibility is given in [4]. See also Remark 3.1, below.) If $\eta_1^2 = \eta_2^2$ and $\rho_1 = \rho_2$ then D_1 and D_2 are said to be equivalent; if $(\lambda_0^{(i)},\lambda_1^{(i)})$ are the parameters associated with D_i $(i = 1,2)$, then D_1 and D_2 are equivalent if and only if $\lambda_0^{(1)} = \lambda_0^{(2)}$ and $\lambda_1^{(1)} = \lambda_1^{(2)}$.

To construct BTIB designs for given (p,k) and any b we next introduce the concept of a generator design. For given (p,k) a generator design is a BTIB design no proper subset of whose blocks forms a BTIB design. The candidates for an optimal design for given (p,k,b) and specified d/σ will be all of the admissible BTIB designs that can be constructed for given b from the list of all generator designs for given (p,k). If two or more equivalent admissible BTIB are available, then it is necessary to consider only one of them (for the purposes of solving the optimization problem (3.1), below); thus in general for given (p,k) we need to consider only the nonequivalent generator designs in our search for an optimal design. For example, for $(p,k) = (4,3)$ the designs

$$D_1 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{matrix}\right\}, \quad D_2 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}$$

with $\lambda_0^{(1)} = \lambda_0^{(2)} = 2$, $\lambda_1^{(1)} = \lambda_1^{(2)} = 2$ are equivalent generator designs, but only one of them need be considered in the optimization problem. As a second example, for $(p,k) = (4,3)$ the design

$$D_3 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 2 & 4 & 4 & 4 \\ 2 & 3 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\}$$

is a generator design with $\lambda_0^{(3)} = 2$, $\lambda_1^{(3)} = 2$ which is equivalent to the union of the two generator designs

$$D_4 = \left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}, \quad D_5 = \left\{\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{matrix}\right\}$$

with $\lambda_0^{(4)} = 2$, $\lambda_1^{(4)} = 0$, $\lambda_0^{(5)} = 0$, $\lambda_1^{(5)} = 2$, respectively; thus it suffices to consider the generator designs D_4 and D_5 in the optimization problem.

<u>Remark 3.1</u>: We note that $(\lambda_0^{(1)},\lambda_1^{(1)}) = (2,2)$ and $(\lambda_0^{(4)} + \lambda_0^{(5)}, \lambda_1^{(4)} + \lambda_1^{(5)}) = (2,2)$ for the BTIB designs D_1 and $D_4 \cup D_5$, respectively. Hence $\mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\}$ of (2.2) is the same for D_1 and $D_4 \cup D_5$ (since $N = kb$ and hence n^2/N in (2.2) is the same for both designs); also, ρ of (2.4) is the same for both designs. This implies that the probability (2.7) is the same for both designs even though D_1 requires $b = 7$ and $D_4 \cup D_5$ requires $b = 8$. We can therefore regard D_1 as <u>dominating</u> $D_4 \cup D_5$ since D_1 requires a <u>smaller</u> number of blocks than $D_4 \cup D_5$ to achieve the <u>same</u> probability; in this sense $D_4 \cup D_5$ is <u>inadmissible</u> with respect to D_1. This more general concept of inadmissibility which allows the comparison of designs with <u>different</u> b-values is introduced and discussed in detail in [4]. The possibility of a design with a smaller b-value dominating a design with a larger b-value does not arise for the cases $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$ considered in detail in the present paper. The concept of strong inadmissibility introduced in the following paragraph also is generalized in [4] to cover situations in which BTIB designs with different b-values are being compared.

There are certain generator designs which will <u>always</u> yield inadmissible BTIB designs, and we would like to eliminate such designs in our search for an optimal design; we call such designs <u>strongly</u> (S-) <u>inadmissible</u> designs. If for given (p,k) we have two designs D_1, D_2 (not

necessarily generator designs) with $b_1 = b_2$, we say that D_2 is <u>S-inadmissible</u> with respect to D_1 if D_2 is inadmissible with respect to D_1, and for <u>any</u> arbitrary BTIB design D_3 we have that $D_2 \cup D_3$ is <u>inadmissible</u> with respect to $D_1 \cup D_3$. S-inadmissibility implies inadmissibility but not conversely; for example, for $(p,k) = (4,3)$ consider the designs

$$D_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 3 & 4 & 4 & 4 \end{Bmatrix}, \quad D_2 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{Bmatrix},$$

with $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 2$, $\lambda_0^{(2)} = 4$, $\lambda_1^{(2)} = 0$; designs D_1 and D_2 are each unions of generator designs, and it is easy to verify that D_2 is <u>inadmissible</u> with respect to D_1. However, $D_2 \cup D_3$ is <u>admissible</u> with respect to $D_1 \cup D_3$ where

$$D_3 = \begin{Bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{Bmatrix},$$

and hence D_2 is not S-inadmissible with respect to D_1. A sufficient condition for a design D_2 to be S-inadmissible with respect to a design D_1 is that $\lambda_0^{(1)} = \lambda_0^{(2)}$ and $\lambda_1^{(1)} > \lambda_1^{(2)}$. (See Remark 5.1 of [3].) Thus for the generator designs

$$D_1 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_2 = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{Bmatrix},$$

with $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 1$, $\lambda_0^{(2)} = 2$, $\lambda_1^{(2)} = 0$, respectively, we see that D_2 is S-inadmissible with respect to D_1.

3.2 Exact optimization problem

We now assume that for given (p,k) there are n generator designs D_i $(1 \leq i \leq n)$ no two of which are equivalent, and no one of which is equivalent to the union of two or more generator designs; we further assume that no D_i $(1 \leq i \leq n)$ is S-inadmissible. Let $\lambda_0^{(i)}, \lambda_1^{(i)}$ be the design parameters associated with D_i, and let b_i be the number of blocks required by D_i $(1 \leq i \leq n)$. Then a BTIB design $D = \bigcup_{i=1}^{n} f_i D_i$ obtained by forming unions of $f_i \geq 0$ replications of D_i has the design parameters $\lambda_0 = \sum_{i=1}^{n} f_i \lambda_0^{(i)}$ and $\lambda_1 = \sum_{i=1}^{n} f_i \lambda_1^{(i)}$ and requires $b = \sum_{i=1}^{n} f_i b_i$ blocks. We shall consider only <u>implementable</u> D, i.e., those for which $\lambda_0 > 0$. It should be noted that for given $\underset{\sim}{D} = (D_1,\ldots,D_n)$ the design D is completely determined by its <u>frequency vector</u> $\underset{\sim}{f} = (f_1,\ldots,f_n)$.

In this setup the integral expression (2.5) for the confidence coefficient can be regarded as a function of $\underset{\sim}{f}$ for given $\underset{\sim}{D}$, p, k, $b = \sum_{i=1}^{n} f_i b_i$ and for specified $\xi = d\sqrt{kb}/\sigma$. We therefore denote (2.7) by $g_E(\underset{\sim}{f} | \underset{\sim}{D}; p,k,b; \xi) = g_E$ (say) where the subscript E stands for <u>exact</u>. Our optimization problem can be stated as:

For given (p,k,b), generator designs $\underset{\sim}{D}$, and specified $\xi = d\sqrt{kb}/\sigma > 0$ choose $\underset{\sim}{f}$ so as to

$$\begin{aligned} &\text{maximize} \quad g_E(\underset{\sim}{f}|\underset{\sim}{D};\ p,k,b;\ \xi) \\ &\text{subject to} \quad \sum_{i=1}^{n} f_i b_i = b, \\ &\qquad\qquad \sum_{i=1}^{n} f_i \lambda_0^{(i)} > 0, \\ &\qquad\qquad f_i \geq 0 \quad (1 \leq i \leq n). \end{aligned} \tag{3.1}$$

Sometimes a dual of (3.1) might be of interest. This happens when b is at the disposal of the experimenter, and he wishes to guarantee a specified confidence coefficient using the smallest possible b. In this situation our optimization problem can be stated as:

For given (p,k), generator designs $\underset{\sim}{D}$, and specified $d/\sigma > 0$ and α $(0 < \alpha < 1)$ choose $\underset{\sim}{f}$ so as to

$$\begin{aligned} &\text{minimize} \quad b = \sum_{i=1}^{n} f_i b_i \\ &\text{subject to} \quad g_E(\underset{\sim}{f}|\underset{\sim}{D};\ p,k,b;\ \xi) \geq 1-\alpha, \\ &\qquad\qquad \sum_{i=1}^{n} f_i \lambda_0^{(i)} > 0 \\ &\qquad\qquad f_i \geq 0 \quad (1 \leq i \leq n). \end{aligned} \tag{3.2}$$

We provide numerical solutions to (3.1) for selected (p,k,b), d/σ and to (3.2) for selected $(p,k,1-\alpha)$, d/σ in Appendix 3.

3.3 Simplification of the optimization problem for $p \geq 2$, $k = 2$ and for $p = 3$, $k = 3$.

In order to solve (3.1) it is first necessary to list all generator designs $D_1,\ldots,D_n$ for given (p,k) and then to enumerate all frequency vectors $\underset{\sim}{f} = (f_1,\ldots,f_n)$ associated with admissible designs $D = \bigcup_{i=1}^{n} f_i D_i$ for given $b = \sum_{i=1}^{n} f_i b_i$. Even for small values of (p,k) such as $p = 4$, $k = 3$ there are surprisingly many generator designs, and the number n of such designs is presently unknown to us. (It is known that n is finite for any (p,k).) We have developed several different methods of constructing generator designs (these being described in Section 3.2 of [3]) but it is known that such methods do not yield all possible generator designs for given (p,k).

However, for $p \geq 2$, $k = 2$ and for $p = 3$, $k = 3$ it turns out that essentially only two generator designs need be considered for each such (p,k). For $p \geq 2$, $k = 2$ this is obvious since the only two generator designs possible are

$$D_0 = \begin{Bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & & p \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & \cdots & p-1 \\ 2 & 3 & & p \end{Bmatrix}. \tag{3.3}$$

(In (3.3) and (3.4), and from Section 3.3 on, we use the notation D_0 to denote a generator design containing zeros, and D_1 to denote a generator design containing no zeros.) For $p = 3$, $k = 3$ we must

consider only the two generator designs

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}; \tag{3.4}$$

all other generator designs are either S-inadmissible w.r.t. D_0 for $b = 3$, or are equivalent to $\bigcup_{i=0}^{1} f_i D_i$ for $b = \sum_{i=0}^{1} f_i b_i > 3$. For example

$$D_0^{(1)} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{Bmatrix}, \quad D_0^{(2)} = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{Bmatrix}, \quad D_0^{(3)} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 3 \end{Bmatrix} \tag{3.5}$$

are generator designs which are S-inadmissible w.r.t. D_0. Also

$$D_0^{(4)} = \begin{Bmatrix} 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 3 & 3 & 3 \end{Bmatrix}, \quad D_0^{(5)} = \begin{Bmatrix} 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 & 3 \\ 2 & 2 & 3 & 3 & 3 \end{Bmatrix} \tag{3.6}$$

are generator designs which are equivalent to $D_0 \cup 2D_1$ for $b = 5$; therefore any BTIB design containing $D_0^{(4)}$ or $D_0^{(5)}$ (or any other equivalent design for $b = 5$) can be replaced by an equivalent design employing D_0 and D_1 in place of $D_0^{(4)}$ or $D_0^{(5)}$. Thus we can limit consideration to D_0 and D_1 of (3.4) to construct the admissible designs for $p = 3$, $k = 3$.

3.4 An admissibility result for the case of two generator designs

For given (p,k) suppose that we have only two generator designs D_0, D_1 to consider as in (3.3) and (3.4) for $p \geq 2$, $k = 2$ and $p = 3$, $k = 3$, respectively. Let $\lambda_0^{(i)}$, $\lambda_1^{(i)}$ be the parameters associated with D_i, and let b_i be the number of blocks required by D_i $(i = 1,2)$; note that $\lambda_0^{(1)} = 0$. For fixed b consider all possible BTIB designs $D = f_0 D_0 \cup f_1 D_1$ $(f_0 \geq 1)$ with $\sum_{i=0}^{1} f_i b_i = b$. For such D we now state a general theorem which enables us to rule out as inadmissible certain designs with sufficiently "large" f_0 ("small" f_1). We first note that for such D we can express η^2 and ρ as

$$\eta^2(f_0) = \frac{k^2 b\{f_0(b_1\lambda_0^{(0)}+b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b\lambda_1^{(1)}\}}{f_0\lambda_0^{(0)}\{f_0(b_1\lambda_0^{(0)} + p[b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}]) + pb\lambda_1^{(1)}\}} \qquad (3.7)$$

$$\rho(f_0) = \frac{f_0(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b\lambda_1^{(1)}}{f_0(b_1\lambda_0^{(0)}+b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b\lambda_1^{(1)}} . \qquad (3.8)$$

Theorem 3.1: Let (p,k,b) and (D_0,D_1) be given. For these D_0, D_1 define

$$C = (p-1)b_0\lambda_0^{(0)}\lambda_1^{(1)} - b_1(\lambda_0^{(0)}+p\lambda_1^{(0)})(\lambda_0^{(0)}+\lambda_1^{(0)}). \qquad (3.9)$$

Consider the class of all designs $D = f_0 D_0 \cup f_1 D_1$ $(f_0 \geq 1,\ f_1 \geq 0)$ satisfying $\sum_{i=0}^{1} f_i b_i = b$. Let f_0^U denote the upper bound on f_0. Then we have either of the following two cases:

Case 1: If $C > 0$ then there exists an integer $f_0^L \geq 2$ which is the smallest value of f_0 satisfying $\eta^2(f_0) \geq \eta^2(f_0-a)$ where a is the smallest positive integer such that b_1 divides (ab_0). If $f_0^L \leq f_0^U$ then all designs D with $f_0 \geq f_0^L$ are inadmissible; i.e., $f_0^L - 1$ is the largest admissible value of f_0.

Case 2: If $C \leq 0$ then all designs D are admissible.

The above result follows from the fact that η^2 is a quasiconvex function of f_0, and ρ is a strictly decreasing function of f_0. The proof of the theorem is given in Appendix 1. If Case 1 holds, in order that $f_0^L \leq f_0^U$ it is necessary that b be sufficiently large; as b increases, the number of designs eliminated as being inadmissible increases.

4. EXACT (DISCRETE) OPTIMAL DESIGNS

4.1 General results for $p \geq 2$, $k = 2$

For $p \geq 2$, $k = 2$ any BTIB design D can be written as $f_0D_0 \cup f_1D_1$ where $f_0 \geq 1$, $f_1 \geq 0$ and D_0, D_1 are given by (3.3). For D_0 we have $\lambda_0^{(0)} = 1$, $\lambda_1^{(0)} = 0$, $b_0 = p$; for D_1 we have $\lambda_0^{(1)} = 0$, $\lambda_1^{(1)} = 1$ $b_1 = p(p-1)/2$. Thus (see (3.9)) $C = p(p-1)/2 > 0$, and Case 1 holds for all $p \geq 2$.

General expressions for η^2 and ρ are

$$\eta^2(f_0) = \frac{4b\{2b + p(p-3)f_0\}}{pf_0\{2b - (p+1)f_0\}} \tag{4.1a}$$

$$\rho(f_0) = \frac{2(b-pf_0)}{2b + p(p-3)f_0} \tag{4.1b}$$

where

$$b = pf_0 + p(p-1)f_1/2. \tag{4.1c}$$

The critical number f_0^L is the smallest f_0 satisfying (4.1c) for which $\eta^2(f_0) \geq \eta^2(f_0 - a)$ where $a = p-1$ if p is even, and $a = (p-1)/2$ if p is odd; thus we have

$$f_0^L = \begin{cases} \operatorname{int}\left[\dfrac{2b+1}{2} - \sqrt{\dfrac{b^2}{3} + \dfrac{1}{4}}\right] & \text{for } p = 2 \quad (4.2a) \\[2ex] \operatorname{int}\left[\dfrac{b+2}{4}\right] & \text{for } p = 3 \quad (4.2b) \\[2ex] \operatorname{int}\left[\dfrac{p(p-3)a - 4b}{2p(p-3)} + \sqrt{\dfrac{4(p-1)^2 b^2}{(p+1)p^2(p-3)^2} + \dfrac{a^2}{4}}\right] & \text{for } p \geq 4 \quad (4.2c) \end{cases}$$

where $\operatorname{int}[z]$ denotes the smallest integer $\geq z$.

We now consider in detail the special case $p = 2$, $k = 2$ in order to illustrate our approach. The same approach is used for the other (p,k) values.

4.1.1 Results for $p = 2$, $k = 2$

For $p = 2$, $k = 2$ all designs $D = f_0 D_0 \cup f_1 D_1$ are generated from D_0, D_1 (of (3.3)) for arbitrary $b = 2f_0 + f_1$ ($b = 2,3,\ldots$) where $1 \leq f_0 \leq b/2$, $0 \leq f_1 \leq b-2$ for b even, and $1 \leq f_0 \leq (b-1)/2$, $1 \leq f_1 \leq b-2$ for b odd. Equation (4.2a) gives f_0^L while $f_0^U = b/2$ or $(b-1)/2$ according as b is even or odd, all inadmissible f_0 satisfying $f_0^L \leq f_0 \leq f_0^U$. Thus for $b \leq 5$ all f_0 are admissible; for $b = 6$, $(f_0,f_1) = (3,0)$ is inadmissible; for $b = 20$, $(f_0,f_1) = (9,2)$

and (10,0) are inadmissible, etc. In Table 4.1 we have enumerated all designs for $b = 2(1)13$, and have given the associated η^2 and ρ; the inadmissible designs are noted.

Conceptually one then proceeds as follows: We are given (p,k,b), and d/σ is specified. For the given b we prepare a table of all BTIB designs, and for each such admissible design compute the associated confidence coefficient as a function of d/σ. This has been done for $b = 13$ in Table 4.2. The optimal design is then the one which maximizes the confidence coefficient for the specified d/σ. We note in Table 4.2 that each admissible design is optimal for one of the tabulated values of d/σ. (We do not claim that it is always true for arbitrary (p,k,b) that each admissible design is optimal for some d/σ.) Note that the design $f_0 = 6$, $f_1 = 1$ is inadmissible. Thus we have obtained a numerical solution to (3.1) for $p = 2$, $k = 2$, $b = 13$. The relevant information in Table 4.2 concerning the optimal designs can be summarized as in Table 4.3; in this table the optimal f_i are denoted by $\hat{f}_i$ $(i = 0,1)$ and the associated maximum confidence coefficient by $\hat{g}_E$. Finally, tables such as Table 4.2 can be prepared for arbitrary $b \geqq 2$, and each of these can be summarized as in Table 4.3. Table A3.1-I in Appendix 3 combines such tables for $b \geqq 2$ and lists for each $b = 2(1)19$ and specified $d/\sigma = 0.2(0.2)2.0$ and for each $b = 20(1)40$ and $d/\sigma = 0.1(0.1)1.0$ the optimal design and associated $\hat{g}_E$.

If it is desired to solve the dual problem (3.2), then the information in Table A3.1-I can be used to obtain the optimal design (minimum b) to achieve a specified confidence coefficient, provided that $b \leqq 40$ solves the problem. If $b > 40$ is required, then the optimal design

Table 4.1

Enumeration of Designs[1/] for p = 2, k = 2 and b = 2(1)13

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

b	f_0	f_1	η^2	ρ	b	f_0	f_1	η^2	ρ
2	1	0	8.00	0.000	10	1	8	21.18	0.889
					10	2	6	11.43	0.750
3	1	1	8.00	0.500	10	3	4	8.48	0.571
					10	4	2	7.50	0.333
4	1	2	9.60	0.667	10	*5	0	8.00	0.000
4	2	0	8.00	0.000					
					11	1	9	23.16	0.900
5	1	3	11.43	0.750	11	2	7	12.38	0.778
5	2	1	7.50	0.333	11	3	5	9.03	0.625
					11	4	3	7.70	0.429
6	1	4	3.33	0.800	11	5	1	7.54	0.167
6	2	2	8.00	0.500					
6	*3	0	8.00	0.000	12	1	10	25.14	0.909
					12	2	8	13.33	0.800
7	1	5	15.27	0.833	12	3	6	9.60	0.667
7	2	3	8.75	0.600	12	4	4	8.00	0.500
7	3	1	7.47	0.250	12	5	2	7.47	0.286
					12	*6	0	8.00	0.000
8	1	6	17.23	0.857					
8	2	4	9.60	0.667	13	1	11	27.13	0.917
8	3	2	7.62	0.400	13	2	9	14.30	0.818
8	*4	0	8.00	0.000	13	3	7	10.20	0.700
					13	4	5	8.36	0.556
9	1	7	19.20	0.875	13	5	3	7.56	0.375
9	2	5	10.50	0.714	13	*6	1	7.58	0.143
9	3	3	8.00	0.500					
9	4	1	7.50	0.200					

[1/] Inadmissible designs are marked with an asterisk(*).

Table 4.2

Confidence Coefficient[1/] as a Function of f_0 and d/σ

for $p = 2$, $k = 2$ when $b = 13$

$\frac{d}{\sigma}$	$\xi = \sqrt{kb}\frac{d}{\sigma}$	$f_0 = 1$	$f_0 = 2$	$f_0 = 3$	$f_0 = 4$	$f_0 = 5$	$f_0 = 6$
		$\eta^2 = 27.13$	$\eta^2 = 14.30$	$\eta^2 = 10.20$	$\eta^2 = 8.36$	$\eta^2 = 7.56$	$\eta^2 = 7.58$
		$\rho = 0.917$	$\rho = 0.818$	$\rho = 0.700$	$\rho = 0.566$	$\rho = 0.375$	$\rho = 0.143$
0.05	0.255	0.4542M	0.4296	0.4057	0.3795	0.3490	0.3109
0.10	0.510	0.4739M	0.4571	0.4385	0.4164	0.3884	0.3509
0.15	0.765	0.4936M	0.4847	0.4718	0.4539	0.4288	0.3926
0.20	1.020	0.5134M	0.5124	0.5053	0.4918	0.4700	0.4354
0.25	1.275	0.5332	0.5400M	0.5388	0.5298	0.5114	0.4787
0.30	1.530	0.5529	0.5675	0.5719M	0.5675	0.5526	0.5222
0.35	1.785	0.5724	0.5946	0.6045M	0.6045	0.5931	0.5653
0.40	2.040	0.5918	0.6213	0.6364	0.6406M	0.6326	0.6074
0.45	2.295	0.6109	0.6474	0.6674	0.6754M	0.6707	0.6482
0.50	2.550	0.6298	0.6728	0.6973	0.7087M	0.7070	0.6871
0.55	2.804	0.6484	0.6974	0.7259	0.7403	0.7413M	0.7240
0.60	3.059	0.6666	0.7212	0.7530	0.7700	0.7733M	0.7584
0.65	3.314	0.6844	0.7440	0.7787	0.7977	0.8028M	0.7902
0.70	3.569	0.7018	0.7658	0.8027	0.8233	0.8299M	0.8193
0.75	3.824	0.7188	0.7866	0.8251	0.8466	0.8544M	0.8457
0.80	4.079	0.7353	0.8062	0.8458	0.8679	0.8764M	0.8692
0.85	4.334	0.7513	0.8246	0.8648	0.8869	0.8959M	0.8901
0.90	4.589	0.7667	0.8420	0.8821	0.9040	0.9130M	0.9084
0.95	4.844	0.7816	0.8581	0.8978	0.9190	0.9279M	0.9242
1.00	5.099	0.7959	0.8731	0.9119	0.9322	0.9407M	0.9379

[1/] The maximum confidence coefficient in each row is marked with an M; the associated f_0 and $f_1 = (13-2f_0)/2$ denoted by $(\hat{f}_0, \hat{f}_1)$, is the optimal design for that value of d/σ when $b = 13$.

Table 4.3

Optimal Design and Associated Confidence Coefficient

as a Function d/σ

for $p = 2$, $k = 2$ when $b = 13$

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad b = 13 = 2f_0 + f_1$$

d/σ	$\hat{f}_0$	$\hat{f}_1$	$\hat{g}_E$
0.05	1	11	0.4542
0.10	1	11	0.4739
0.15	1	11	0.4936
0.20	1	11	0.5134
0.25	2	9	0.5400
0.30	3	7	0.5719
0.35	3	7	0.6045
0.40	4	5	0.6406
0.45	4	5	0.6754
0.50	4	5	0.7087
0.55	5	3	0.7413
0.60	5	3	0.7733
0.65	5	3	0.8028
0.70	5	3	0.8299
0.75	5	3	0.8544
0.80	5	3	0.8764
0.85	5	3	0.8959
0.90	5	3	0.9130
0.95	5	3	0.9279
1.00	5	3	0.9407

for $1-\alpha = 0.80(0.05)0.95, 0.99$ and $d/\sigma = 0.2(0.2)2.0$ can be found in Table A3.1-II. (The entries in this table were obtained by a complete computer search for the optimal b.)

It might be noted that in Table A3.1-I the confidence coefficient is strictly increasing in b for all values of d/σ. That this is not the case for all (p,k) can be seen, e.g., for $p = 4$, $k = 2$ in Table A3.3-I.

4.1.2 Results for $3 \leq p \leq 6$, $k = 2$

The same approach as was adopted to obtain Tables A3.1-I and A3.1-II for $p = 2$, $k = 2$ was also used to obtain corresponding tables for $p = 3(1)6$, $k = 2$. The range of b- and d/σ-values covered in each of these tables is summarized for easy reference in Table 4.4 below.

Table 4.4

List of Tables of Exact (Discrete) Optimal Designs for $k = 2$

p	Optimal Design and $\hat{g}_E$	Optimal Design to achieve $1-\alpha$	Range of b-, d/σ- and $(1-\alpha)$-values in each table
2	Table A3.1-I		$b = 2(1)19$ with $d/\sigma = 0.2(0.2)2.0$ $b = 20(1)40$ with $d/\sigma = 0.1(0.1)1.0$
		Table A3.1-II	$1-\alpha = 0.80(0.05)0.95, 0.99$ $d/\sigma = 0.2(0.2)2.0$
3	Table A3.2-I		$b = 3(3)48$ with $d/\sigma = 0.2(0.2)2.0$ $b = 51(3)114$ with $d/\sigma = 0.1(0.1)1.0$
		Table A3.2-II	$1-\alpha = 0.80(0.05)0.95, 0.99$ $d/\sigma = 0.2(0.2)2.0$
4	Table A3.3-I		$b = 4, 8(2)36$ with $d/\sigma = 0.2(0.2)2.0$ $b = 38(2)78$ with $d/\sigma = 0.1(0.1)1.0$
		Table A3.3-II	$1-\alpha = 0.80(0.05)0.95, 0.99$ $d/\sigma = 0.2(0.2)2.0$
5	Table A3.4-I		$b = 5(5)85$ with $d/\sigma = 0.2(0.2)2.0$ $b = 90(5)200$ with $d/\sigma = 0.1(0.1)1.0$
		Table A3.4-II	$1-\alpha = 0.80(0.05)0.95, 0.99$ $d/\sigma = 0.2(0.2)2.0$
6	Table A3.5-I		$b = 6(6)18(3)57$ with $d/\sigma = 0.2(0.2)2.0$ $b = 60(3)123$ with $d/\sigma = 0.1(0.1)1.0$
		Table A3.5-II	$1-\alpha = 0.80(0.05)0.95, 0.99$ $d/\sigma = 0.2(0.2)2.0$

It sometimes occurs that $\hat{g}_E$ decreases as b increases for particular values of d/σ (see, e.g., Table A3.3-I); when this occurs the experimenter should, of course, use the smallest value of b which guarantees an acceptable value of $1-\alpha$. It should also be noted that for fixed d/σ and b sufficiently large, the confidence coefficient $\hat{g}_E$ is strictly increasing in b.

4.2 Results for $p = 3$, $k = 3$

As discussed in Section 3.3, for $p = 3$, $k = 3$ we can limit consideration to BTIB designs D which can be written as $f_0 D_0 \cup f_1 D_1$ where $f_0 \geqq 1$, $f_1 \geqq 0$ and D_0, D_1 are given by (3.4) for arbitrary $b = 3f_0 + f_1$ $(b = 3,4,\ldots)$; here $1 \leqq f_0 \leqq b/3$, $0 \leqq f_1 \leqq b-3$. For D_0 we have $\lambda_0^{(0)} = 2$, $\lambda_1^{(0)} = 1$, $b_0 = 3$; for D_1 we have $\lambda_0^{(1)} = 0$, $\lambda_1^{(1)} = 1$, $b_1 = 1$. Thus (see (3.9)) $C = -3 < 0$, and Case 2 holds. Therefore <u>all</u> designs D are admissible.

For $p = 3$, $k = 3$ expressions for η^2 and ρ are

$$\eta^2(f_0) = \frac{9b^2}{2f_0(3b-4f_0)} \tag{4.3a}$$

$$\rho(f_0) = \frac{b - 2f_0}{b} \tag{4.3b}$$

where

$$b = 3f_0 + f_1. \tag{4.3c}$$

Table A3.6-I lists for each $b = 3(1)18$ and $d/\sigma = 0.2(0.2)2.0$ and for each $b = 19(1)40$ and $d/\sigma = 0.1(0.1)1.0$ the optimal design and

associated $\hat{g}_E$; Table A3.6-II lists the optimal design for $1-\alpha = 0.80(0.05)0.95, 0.99$ and $d/\sigma = 0.2(0.2)2.0$.

5. APPROXIMATE (CONTINUOUS) OPTIMAL DESIGNS

5.1 Preliminaries

As in Section 3.3 we continue to deal with situations in which it suffices to consider only two generator designs. In Section 4 we noted that the number of competing admissible designs in general (but not always) increases with b for fixed (p,k). We have also seen that for each (p,k,b) the optimal design depends on d/σ. Thus a separate determination of the optimal design must be made for each (p,k,b), d/σ combination. This represents a formidable computing and tabulation task. The solutions for many of the most useful combinations are given in Appendix 3.

In order to extend the results given in Appendix 3, and to do so in a compact form we introduce in this section an approximation (for large b) to the discrete optimization problem considered in Section 4, i.e., we shall replace the discrete optimization problems (3.1) and (3.2) by their continuous analogues. The continuous versions are analytically more tractable and computationally easier to solve. Moreover, since their solution does not depend individually on b and d/σ but only on these quantities through $\xi = d\sqrt{kb}/\sigma$, the number of solutions that must be tabulated is drastically reduced. The approximate continuous optimal solutions (discussed in Section 5.3 below) are given in Tables A4.1-I through A4.2-II; in Section 5.5 the accuracy of these solutions is checked against the corresponding exact discrete

optimal solutions for $p = 2$, $k = 2$, $d/\sigma = 0.2$ and selected b, and the agreement is shown to be excellent.

5.2 Approximate (continuous) optimal designs for given (p,k), (D_0,D_1) and specified ξ

5.2.1 Definition of γ

For fixed (p,k,b) and (D_0,D_1) we define

$$\gamma = \frac{f_0 b_0}{b} = \frac{f_0 b_0}{f_0 b_0 + f_1 b_1} \tag{5.1}$$

which is the proportion of the total number of blocks allocated to D_0. For large values of b we shall treat $\gamma \in (0,1]$ as a nonnegative continuous variable. Then regarding η^2 and ρ of (2.3) and (2.4), respectively, as continuous functions of γ, we consider the integral (2.7) as a function of γ for each (p,k,ξ)-combination, and denote its value by $g_A(\gamma|D_0,D_1; p,k; \xi) = g_A$ (say) where the subscript A stands for approximate.

5.2.2 Optimization with respect to γ

For given values of (p,k) and specified ξ we shall maximize g_A w.r.t. γ for fixed (D_0,D_1); the value $\hat{\gamma}$ (say) of γ at which the maximum is attained will be termed the continuous optimal design. We seek to obtain $\hat{\gamma}$ as the solution in γ of the equation $\partial g_A/\partial\gamma = 0$. In doing so we must be assured that an allowable solution in γ exists, that it is unique, and that it is in fact associated with the maximum of g_A. Actually it turns out that either a unique solution in γ of $\partial g_A/\partial\gamma = 0$ exists, lies in the interval $(0,1)$, and corresponds to the

maximizing value, or that the maximum value of g_A for $\gamma \in [0,1]$ occurs at the boundary $\gamma = 0$ for ξ sufficiently small or at the boundary $\gamma = 1$ for ξ sufficiently large; the solution $\gamma = 1$ occurs only under Case 2 (cf. Theorem 3.1).

We show in Appendix 2 that

$$\frac{\partial g_A}{\partial \gamma} = \frac{p\phi(\xi/\eta)}{2\eta^2} h(\gamma | D_0, D_1;\ p,k;\ \xi) \tag{5.2}$$

where

$$h(\gamma | D_0, D_1;\ p,k;\ \xi)$$

$$= \frac{(p-1)\eta^2}{(1-\rho^2)^{1/2}} \frac{\partial \rho}{\partial \gamma} \phi\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\right] \Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}} \,\middle|\, \frac{\rho}{1+2\rho}\right]$$

$$- \frac{\xi}{\eta}\frac{\partial \eta^2}{\partial \gamma} \Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}} \,\middle|\, \frac{\rho}{1+\rho}\right]. \tag{5.3}$$

In (5.3), $\phi(\cdot)$ denotes the standard normal pdf and $\Phi_r(x|\tau)$ denotes the cdf at the equicoordinate point x of an r-variate equicorrelated standard normal distribution with common correlation coefficient τ; the quantities η^2, ρ, $\partial\eta^2/\partial\gamma$ and $\partial\rho/\partial\gamma$ are given as functions of γ by (A.1)-(A.4), respectively, in Appendix 1. Note that the sign of $\partial g_A/\partial\gamma$ depends only on the sign of $h(\gamma | D_0, D_1;\ p,k;\ \xi)$.

5.2.3 Study of g_A and its maximum

In this section we shall study the behavior of g_A as a function of γ and ξ for fixed (p,k) and (D_0, D_1). It is straightforward to check that in the limiting case $\gamma = 0$ we have $\eta^2 = \infty$, $\rho = 1$ and

hence $g_A = 1/2$. For fixed $\gamma > 0$, we note that as ξ increases from 0 to ∞, g_A increases from $\Phi_p(0|\rho)$ to 1.

All of our calculations and studies lead us to conclude that g_A regarded as a function of γ attains a unique maximum at $\hat{\gamma}$, the value of which depends on ξ and (D_0, D_1). For all ξ $(0 \leqq \xi \leqq \xi_0 = \xi_0(D_1; p,k))$ where ξ_0 is given by (5.6) below, we have that g_A is strictly decreasing in γ, and hence $\hat{\gamma} = 0$ maximizes g_A, the maximum being equal to 1/2. This result parallels those obtained in [1] and [2].

To study the behavior of g_A as a function of γ for $\xi > \xi_0$ we note that for large ξ the second term in (5.3) dominates and hence for large ξ we have $\mathrm{sgn}(\partial g_A/\partial\gamma) = \mathrm{sgn}(h) = -\mathrm{sgn}(\partial\eta^2/\partial\gamma)$. In Appendix 1 we have shown that η^2 is a quasiconvex function of γ for $0 < \gamma \leqq 1$. Depending on whether η^2 achieves a minimum in the interior of $[0,1]$ or at the boundary $\gamma = 1$ we have the following two cases which depend on C of (3.9):

<u>Case 1 ($C > 0$)</u>: In this case g_A has a unique maximum at $\hat{\gamma}$ $(0 < \hat{\gamma} < 1)$ for all $\xi > \xi_0$; here $\hat{\gamma}$ is the unique solution in γ of the equation

$$h(\gamma|D_0, D_1; p,k; \xi) = 0. \qquad (5.4)$$

The maximizing solution $\hat{\gamma}$ is a strictly increasing function of ξ for $\xi > \xi_0$. In the limit $(\xi \to \infty)$ the maximizing solution approaches γ^L where

$$\gamma^L = \lim_{b\to\infty} \frac{b_0 f_0^L}{b}$$

which is the <u>largest admissible value</u> of γ. This limiting value γ^L can

also be easily found by solving the quadratic equation $\partial\eta^2/\partial\gamma = 0$ in γ, i.e., g_A is maximized for $\xi \to \infty$ (which causes the confidence coefficient $(1-\alpha)$ to approach unity) by minimizing the common variance of the $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

Case 2 $(C \leq 0)$: In this case there exists a positive constant $\xi_1 = \xi_1(D_0,D_1;p,k)$, $(0 < \xi_0 < \xi_1 < \infty)$ such that for all $\xi \in (\xi_0,\xi_1)$, g_A has a unique maximum at $\hat{\gamma} < 1$; here $\hat{\gamma}$ is the unique solution in γ to (5.4). The maximizing solution $\hat{\gamma}$ is a strictly increasing function of ξ for $\xi \in (\xi_0,\xi_1)$ with $\hat{\gamma} \to 1$ as $\xi \to \xi_1$. For all $\xi \geq \xi_1$ the maximizing solution is also $\hat{\gamma} = 1$ (which implies no replications of D_1). As with Case 1, in the limit $(\xi \to \infty)$, the maximizing solution is the value of γ which minimizes the common variance of the $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

5.2.4 Definition of ξ_0

As the first step in finding a closed expression for ξ_0 we consider $\lim_{\gamma\to 0} h(\gamma|D_0,D_1;\ p,k;\ \xi)$. From (A.3), (A.4) and (A.5) we see that $\lim_{\gamma\to 0} \frac{\partial\rho}{\partial\gamma} < 0$ and $\lim_{\gamma\to 0} \frac{\partial\eta^2}{\partial\gamma} = -\infty < 0$. Therefore

$$\lim_{\gamma\to 0} h(\gamma|D_0,D_1;\ p,k;\ \xi)\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}0 \iff \xi\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\xi_0$$

where $\xi_0 = \xi_0(D_1;\ p,k)$ is defined by

$$\xi_0 = \lim_{\gamma\to 0} \frac{(p-1)\eta^3\,\frac{\partial\rho}{\partial\gamma}}{(1-\rho^2)^{1/2}\,\frac{\partial\eta^2}{\partial\gamma}} \left\{ \frac{\phi\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\right]\Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}}\,\middle|\,\frac{\rho}{1+2\rho}\right]}{\Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\,\middle|\,\frac{\rho}{1+\rho}\right]} \right\}. \quad (5.5)$$

We evaluate this limit in Appendix A2 and show it to be

$$\xi_0 = \frac{1}{2}\, p(p-1)k\Phi_{p-2}(0|1/3)\sqrt{\frac{b_1}{\pi p \lambda_1^{(1)}}}\;. \tag{5.6}$$

Note that ξ_0 depends only on D_1. If D_1 is either a BIB or a RB design between the p test treatments as in (3.3) and (3.4), then we can substitute $\lambda_1^{(1)} = b_1 k(k-1)/p(p-1)$ in (5.6) to obtain

$$\xi_0 = \frac{1}{2}\, p\Phi_{p-2}(0|1/3)\sqrt{\frac{k(p-1)^3}{(k-1)\pi}}\;. \tag{5.7}$$

Remark 5.1: It is known (see, e.g., Gupta [5]) that $\Phi_0(0|1/3) = 1$, $\Phi_1(0|1/3) = 1/2$, $\Phi_2(0|1/3) = 1/4 + (1/2\pi)\arcsin(1/3)$, and $\Phi_3(0|1/3) = 1/8 + (3/4\pi)\arcsin(1/3)$; $\Phi_t(0|1/3)$ has been computed to five decimal places for $t = 1(1)12$ by Gupta [5, Table II, p. 817].

5.2.5 Definition of ξ_1

For Case 2, we define ξ_1 as the smallest value of ξ for which $\hat{\gamma}$ equals unity; hence for $\xi > \xi_1$ we have $\hat{\gamma} = 1$. Thus ξ_1 is the solution in ξ of the equation

$$h(\gamma|D_0,D_1;\, p,k;\, \xi)\Big|_{\gamma=1} = 0.$$

In general ξ_1 depends on D_0, D_1.

5.2.6 Uniqueness of maximum of g_A as a function of γ

As mentioned in Section 5.2.3, we have not yet proved analytically the existence of a unique maximum for g_A as a function of γ when

$\xi_0 \leq \xi < \infty$ (nor did we prove the corresponding result in our previous papers); however, all of our numerical calculations and certain analytical considerations point to this conclusion.

We have computed g_A as a function of γ for selected values of ξ, and given the results in Tables 5.1A and 5.1B to illustrate Cases 1 and 2, respectively. Table 5.1A is for $p = k = 2$ (with generator designs (D_0, D_1) of (3.3)), and Table 5.1B is for $p = k = 3$ (with generator designs (D_0, D_1) of (3.4)); these computations give representative pictures of the behavior of g_A as a function of γ. The behavior of g_A in Table 5.1A is analogous to that of g_A in Figure 1 of [1] in that g_A has a unique maximum at $\hat{\gamma} < \gamma^L$ for $\xi > \xi_0$. However, unlike the situation in Figure 1 where $\lim_{\gamma \to 1} g_A = 1/2^p$ we now have $\lim_{\gamma \to 1} g_A$ depending on ξ and other parameters of the design.

5.3 Approximate (continuous) optimal designs for given (p,k), (D_0, D_1) and specified confidence coefficient

As pointed out in Section 3 (see (3.2)), sometimes the experimenter wishes to guarantee a specified confidence coefficient $1-\alpha$ for given (p,k), (D_0, D_1) using the smallest possible b, or in the continuous case using the smallest possible ξ, say $\hat{\xi}$. This dual problem can be solved in the continuous case in the following way:

a) If Case 1 holds or if Case 2 holds but $\hat{\xi} < \xi_1$ (see comments below), then solve simultaneously the two equations

$$h(\gamma | D_0, D_1; p, k; \xi) = 0 \tag{5.8}$$

$$\int_{-\infty}^{\infty} \Phi^p \left[\frac{x\sqrt{\rho} + \xi/\eta}{\sqrt{1-\rho}} \right] d\Phi(x) = 1-\alpha \tag{5.9}$$

Table 5.1A

Values of g_A as a Function of γ for Selected ξ

when $D = f_0 D_0 \cup f_1 D_1$ is used for $p = k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}; \quad \text{Case 1: } \xi_0 = 0.7979$$

ξ	γ										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.5	.5000	.4794	.4680	.4569	.4451	.4321	.4174	.4004	.3802	.3558	.3251
2.0	.5000	.5731	.5993	.6161	.6272	.6334	.6352	.6321	.6231	.6063	.5780

Table 5.1B

Values of g_A as a Function of γ for Selected ξ

when $D = f_0 D_0 \cup f_1 D_1$ is used for $p = k = 3$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}; \quad \text{Case 2: } \begin{matrix} \xi_0 = 1.4658 \\ \xi_1 = 4.5081 \end{matrix}$$

ξ	γ										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	.5000	.4707	.4561	.4431	.4305	.4179	.4049	.3914	.3774	.3625	.3468
3.0	.5000	.5879	.6196	.6407	.6556	.6661	.6730	.6769	.6779	.6762	.6716
5.0	.5000	.6978	.7639	.8059	.8350	.8560	.8712	.8822	.8897	.8944	.8965

for γ and ξ, the solutions being $\hat{\gamma}$ and $\hat{\xi}$; here h, η^2 and ρ are given by (5.3), (A.1) and (A.2), respectively.

b) If Case 2 holds and $\hat{\xi} \geq \xi_1$, then solve only (5.9) for ξ with $\gamma = 1$, the solution being $\hat{\gamma} = 1$, $\hat{\xi}$.

To see whether or not $\hat{\xi} < \xi_1$ for $p = k = 3$, (D_0, D_1) given by (3.4), and specified $1-\alpha$ can be found by consulting Table A4.2-I in the Appendix. If $\xi < \xi_1$ for the smallest tabulated $\hat{g}_A \geq 1-\alpha$, then $\hat{\xi} < \xi_1$; if $\xi \geq \xi_1$ for the largest tabulated $\hat{g}_A \leq 1-\alpha$, then $\hat{\xi} \geq \xi_1$. For the intermediate case in which the ξ-values corresponding to the $\hat{g}_A$-values bracketing the specified $1-\alpha$ are, respectively, $< \xi_1$ and $> \xi_1$, then it is necessary to use trial and error to determine if $\hat{\xi} < \xi_1$ or $\hat{\xi} \geq \xi_1$.

5.4 Use of tables of approximate (continuous) optimal designs for $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$

For given (p,k) and (D_0, D_1) tables of approximate continuous optimal designs can be computed using the methods described in Section 5.3. This has been done for $p = 2(1)6$, $k = 2$ for (D_0, D_1) given by (3.3) and for $p = k = 3$ for (D_0, D_1) given by (3.4). The results are summarized in Tables A4.1-I and A4.2-I, respectively, which for selected ξ give $\hat{\gamma}$ and $\hat{g}_A$, and in Table A4.1-II and A4.2-II which for $1-\alpha = 0.75(0.05)0.95$, 0.99 give $\hat{\gamma}$ and $\hat{\xi}$.

Tables A4.1-I and A4.2-I are to be used for the large values of b not considered in Tables A3.1-I through A3.6-I. Here (p,k,b) and σ^2 are given, and (p,k) determines (D_0, D_1). The experimenter specifies d

(his "yardstick") which determines $\xi = d\sqrt{kb}/\sigma$. Entering the appropriate table with (p,k,ξ) the experimenter obtains $\hat{\gamma}$ and the associated $\hat{g}_A$. Then using the relation $\hat{\gamma} \sim \hat{f}_0 b_0/b$, he adopts $\hat{f}_0 = b\hat{\gamma}/b_0$ (to the nearest integer) as his (approximately optimal) design which will guarantee a confidence coefficient of approximately $\hat{g}_A$.

Tables A4.1-II and A4.2-II are to be used for the large values of b not considered in Tables A3.1-II through A3.6-II. Here (p,k) and σ^2 are given, and (p,k) determines (D_0,D_1). The experimenter specifies d and $1-\alpha$. Entering the table with $(p,k,1-\alpha)$ the experimenter obtains $\hat{\gamma}$ and the associated $\hat{\xi}$. Then since $\xi = d\sqrt{kb}/\sigma$ he computes $\hat{b} = \text{int}[(\hat{\xi}\sigma/d)^2/k]$. Finally, he chooses $\hat{f}_0$ so that $b_0\hat{f}_0/\hat{b} \sim \hat{\gamma}$ <u>and</u> $b_0\hat{f}_0 + b_1\hat{f}_1$ is as close as possible $(\geq)$ $\hat{b}$. This yields an approximately optimal design with associated confidence coefficient of approximately $1-\alpha$.

The approximations referred to in the paragraphs above arise because we use a discrete design which is as "close as possible" to the optimal continuous design. These approximations become increasingly more accurate as b increases. The goodness of the approximation is discussed in the next section.

5.5 Comparison of exact and approximate optimal designs

To illustrate the accuracy of the approximation provided by the continuous optimal designs we computed the exact discrete optimal design and the approximate continuous optimal design for $p = 2$, $k = 2$, $d/\sigma = 0.2$ and selected values of b (and thus ξ). The results are displayed in Table 5.2. In this table we have denoted $b_0\hat{f}_0/b$ by $\hat{\gamma}_E$ where $\hat{f}_0$ corresponds to the exact optimal design. We note that the approximate

Table 5.2

Comparison of Exact and Approximate Optimal Designs

($p = 2$, $k = 2$, $d/\sigma = 0.2$)

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

b	ξ	Exact Optimal Design			Approximate Optimal Design	
		f_0	$\hat{\gamma}_E$	$\hat{g}_E$	$\hat{\gamma}$	$\hat{g}_A$
10	0.8944	1	0.2000	0.5028	0.1001	0.5041
12	0.9798	1	0.1667	0.5104	0.1737	0.5104
15	1.0955	2	0.2667	0.5210	0.2567	0.5210
20	1.2649	4	0.4000	0.5390	0.3528	0.5393
25	1.4142	5	0.4000	0.5572	0.4195	0.5572
30	1.5492	7	0.4667	0.5744	0.4693	0.5744
35	1.6733	9	0.5143	0.5908	0.5083	0.5908
40	1.7889	11	0.5500	0.6063	0.5398	0.6064
50	2.0000	15	0.6000	0.6352	0.5881	0.6352
75	2.4495	25	0.6667	0.6965	0.6627	0.6965
100	2.8284	35	0.7000	0.7457	0.7062	0.7457

continuous optimal designs provide excellent approximations to the exact optimal designs even for small values of ξ (associated with "low" joint confidence coefficients), and that $\hat{g}_E$ is only slightly smaller than $\hat{g}_A$; as is to be expected, the approximation improves with increasing ξ.

6. ACKNOWLEDGMENT

The authors are pleased to acknowledge with thanks the efforts of Mr. Carl Emont who spent long hours computing the tables of (discrete) optimal designs given in Appendix 3.

APPENDIX 1

Proof of Theorem 3.1

For mathematical convenience and without loss of generality we shall regard $\gamma = f_0 b_0 / b$ as a continuous variable taking values in the interval $(0,1]$. (We use the same continuous approximation in Section 5.) For $p \geq 2$, $k = 2$ and $p = 3$, $k = 3$ we substitute $f_0 = b\gamma/b_0$ in (3.7) and (3.8) and obtain

$$\eta^2(\gamma) = \frac{k^2 b_0\{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}}{\lambda_0^{(0)}\gamma\{\gamma[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}} \qquad (A.1)$$

and

$$\rho(\gamma) = \frac{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_0\lambda_1^{(1)}}{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}}, \qquad (A.2)$$

respectively. It follows that

$$\frac{\partial\rho}{\partial\gamma} = -\frac{b_0 b_1 \lambda_0^{(0)}\lambda_1^{(1)}}{\{\gamma(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)} + b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}^2} < 0 \qquad (A.3)$$

for $b_1, \lambda_1^{(1)} > 0$, and therefore ρ is strictly decreasing in γ (and hence in f_0). Next we have

$$\frac{\partial\eta^2}{\partial\gamma} = \frac{k^2 b_0 \psi(\gamma)}{\lambda_0^{(0)}\gamma^2\{\gamma[p(b_1\lambda_1^{(0)} - b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}^2} \qquad (A.4)$$

where

$$\psi(\gamma) = \gamma^2(b_0\lambda_1^{(1)}-b_1\lambda_1^{(0)}-b_1\lambda_0^{(0)})[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}]$$

$$- 2\gamma b_0\lambda_1^{(1)}[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] - p(b_0\lambda_1^{(1)})^2. \quad \text{(A.5)}$$

Since $\lim_{\gamma\to 0} \eta^2 = \infty$, it follows that η^2 must be decreasing, at least in a small neighborhood of $\gamma = 0+$; thus it suffices to show that η^2 has at most one stationary point in $(0,1]$, i.e., that the equation $\psi(\gamma) = 0$ has at most one root in $(0,1]$.

Since the constant term $-p(b_0\lambda_1^{(1)})^2$ in (A.5) is negative, a necessary (but clearly not sufficient) condition for both roots of $\psi(\gamma)$ to be real, positive and distinct is that the coefficients of γ^2 and γ in $\psi(\gamma)$ be negative and positive, respectively, i.e., we require that

$$p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)} < 0.$$

Therefore,

$$\left.\frac{\partial\psi(\gamma)}{\partial\gamma}\right|_{\gamma=1} = -2b_1(\lambda_0^{(0)}+\lambda_1^{(0)})[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] > 0.$$

The latter inequality shows that $\psi(\gamma)$ is increasing at $\gamma = 1$. This together with the fact that $\psi(\gamma) \to -\infty$ as $\gamma \to \pm\infty$ implies that at most one root of $\psi(\gamma)$ must be in $(0,1]$.

The proof of the theorem now follows easily since Case 1 or Case 2 holds depending on whether

$$\operatorname{sgn}\left\{\left.\frac{\partial\eta^2}{\partial\gamma}\right|_{\gamma=1}\right\} = \operatorname{sgn}\left\{\left.\psi(\gamma)\right|_{\gamma=1}\right\} = \operatorname{sgn}\{b_1C\}$$

is > 0 or ≤ 0, respectively, where C is given by (3.9). If Case 1 holds then η^2 first decreases and then increases with f_0 (i.e., γ) while ρ always decreases with f_0. Hence there exists a critical number f_0^L which is the smallest value of f_0 at which η^2 starts increasing (for fixed b). Thus f_0^L is the smallest value of f_0 satisfying $\eta^2(f_0) \geq \eta^2(f_0-a)$ where $\underline{a}$ is the smallest positive integer such that ab_0/b_1 is a positive integer. If $f_0^L \leq f_0^U$, then clearly designs with $f_0 \geq f_0^L$ are inadmissible. If Case 2 holds then since both η^2 and ρ are strictly decreasing in f_0 (i.e., γ), all designs $D = f_0 D_0 \cup f_1 D_1$ with $f_0 > 0$ are admissible.

APPENDIX 2

Derivation of Results in Section 5

I. Evaluation and simplification of $\partial g_A/\partial\gamma$

From (2.7) and the definition of g_A given in Section 5.2.1 we obtain by direct calculation

$$\frac{\partial g_A}{\partial\gamma} = \int_{-\infty}^{\infty} p\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)$$
$$\times\left\{\frac{\eta\sqrt{1-\rho}\left(x\eta'\sqrt{\rho}+\dfrac{x\eta\rho'}{2\sqrt{\rho}}\right)-\left(x\eta\sqrt{\rho}+\xi\right)\left(\eta'\sqrt{1-\rho}-\dfrac{\eta\rho'}{2\sqrt{1-\rho}}\right)}{\eta^2(1-\rho)}\right\}dx. \tag{A.6}$$

In (A.6), $\phi(\cdot)$ denotes the standard normal pdf, $\Phi(\cdot)$ denotes the standard normal cdf, $\eta' = \partial\eta/\partial\gamma = (1/2\eta)\partial\eta^2/\partial\gamma$ where $\partial\eta^2/\partial\gamma$ is given by (A.4), and $\rho' = \partial\rho/\partial\gamma$ is given by (A.3). After some simplification (A.6) can be written as

$$\frac{\partial g_A}{\partial\gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}}\left\{\frac{\eta^2\rho'}{\sqrt{\rho}}\int_{-\infty}^{\infty} x\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)dx\right.$$
$$\left.-\ \xi[2\eta'(1-\rho)-\eta\rho']\int_{-\infty}^{\infty}\Phi^{p-1}\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi\left[\frac{x\eta\sqrt{\rho}+\xi}{\eta\sqrt{1-\rho}}\right]\phi(x)dx\right\}. \tag{A.7}$$

Making the change of variables

$$y = (x\eta\sqrt{\rho}+\xi)/\eta\sqrt{1-\rho}$$

(A.7) can be expressed as

$$\frac{\partial g_A}{\partial \gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}}\left\{\frac{\eta^2\rho'}{R^2\sqrt{\rho}}E_1 - \left[\frac{S\eta^2\rho'}{R^2\sqrt{\rho}} + \frac{\xi}{R}\{2\eta'(1-\rho)-\eta\rho'\}E_2\right]\right\} \quad (A.8)$$

where

$$R = \left(\frac{\rho}{1-\rho}\right)^{1/2}, \quad S = \frac{\xi}{\eta\sqrt{1-\rho}}, \quad (A.9)$$

$$E_1 = \int_{-\infty}^{\infty} y\Phi^{p-1}(y)\phi(y)\phi^*(y)dy \quad (A.10)$$

$$E_2 = \int_{-\infty}^{\infty} \Phi^{p-1}(y)\phi(y)\phi^*(y)dy, \quad (A.11)$$

and $\phi^*(y)$ denotes $\phi((y-S)/R)$. We now evaluate E_1 and E_2. Integrating by parts in E_1 with $U = \Phi^{p-1}(y)\phi^*(y)$ and $dV = y\phi(y)dy$ we obtain

$$E_1 = -\frac{1}{R^2}E_1 + \frac{S}{R^2}E_2 + (p-1)E_3 \quad (A.12)$$

where

$$E_3 = \int_{-\infty}^{\infty} \Phi^{p-2}(y)\phi^2(y)\phi^*(y)dy. \quad (A.13)$$

From (A.12) we have

$$E_1 = \frac{S}{R^2+1}E_2 + \frac{(p-1)R^2}{R^2+1}E_3. \quad (A.14)$$

Substituting (A.14) in (A.8) we obtain

$$\frac{\partial g_A}{\partial \gamma} = \frac{p}{2\eta^2(1-\rho^2)^{3/2}}\left\{-\left[\frac{S\eta^2\rho'}{\sqrt{\rho}(R^2+1)} + \frac{\xi}{R}\{2\eta'(1-\rho)-\eta\rho'\}\right]E_2 + \frac{\eta^2\rho'(p-1)}{\sqrt{\rho}(R^2+1)}E_3\right\}. \tag{A.15}$$

By developments similar to those in [1] we can write

$$E_2 = \frac{R}{\sqrt{R^2+1}}\,\phi\left(\frac{S}{\sqrt{R^2+1}}\right)\Phi_{p-1}\left[\frac{S}{\sqrt{(2R^2+1)(3R^2+1)}}\,\middle|\,\frac{R^2}{R^2+1}\right], \tag{A.16}$$

$$E_3 = \frac{R}{\sqrt{(2R^2+1)2\pi}}\,\phi\left(\sqrt{\frac{2S^2}{2R^2+1}}\right)\Phi_{p-2}\left[\frac{S}{\sqrt{(2R^2+1)(3R^2+1)}}\,\middle|\,\frac{R^2}{3R^2+1}\right]. \tag{A.17}$$

Substituting E_2 and E_3 from (A.16) and (A.17) in (A.15), and replacing R and S by their definitions in (A.9) we obtain

$$\frac{\partial g_A}{\partial \gamma} = \frac{p}{2\eta^2\sqrt{1-\rho}}\left\{-2\xi\eta'\sqrt{1-\rho}\;\phi\left(\frac{\xi}{\eta}\right)\Phi_{p-1}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{1+\rho}}\,\middle|\,\frac{\rho}{1+\rho}\right]\right.$$

$$\left. + \frac{\eta^2\rho'(p-1)}{\sqrt{2\pi(1+\rho)}}\,\phi\left(\frac{\xi}{\eta}\sqrt{\frac{2}{1+\rho}}\right)\Phi_{p-2}\left[\frac{\xi}{\eta}\sqrt{\frac{1-\rho}{(1+\rho)(1+2\rho)}}\,\middle|\,\frac{\rho}{1+2\rho}\right]\right\}$$

$$= \frac{p\phi(\xi/\eta)}{2\eta^2}\,h(\gamma|D_0,D_1;\ p,k;\ \xi)$$

where $h(\gamma|D_0,D_1;\ p,k;\ \xi)$ is given by (5.3).

II. Evaluation of the limit in expression (5.5) for ξ_0.

We note from (A.1) and (A.2) that $\lim_{\gamma\to 0}\eta^2 = \infty$ and $\lim_{\gamma\to 0}\rho = 1$. Since $\Phi_{p-1}(0|1/2) = 1/p$ we have from (5.5) that

$$\xi_0 = \frac{p(p-1)\Phi_{p-2}(0|1/3)}{\sqrt{2\pi}} \lim_{\gamma\to 0}\left\{\frac{\eta^3 \frac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}\,\frac{\partial\eta^2}{\partial\gamma}}\right\}. \tag{A.18}$$

Using (A.3) and (A.4) we write

$$\frac{\eta^3}{\frac{\partial\eta^2}{\partial\gamma}} = \frac{k}{\psi(\gamma)}\sqrt{\frac{b_0\gamma}{\lambda_0^{(0)}}}\;\frac{\{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}^{3/2}}{\{\gamma[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}^{-1/2}} \tag{A.19}$$

where $\psi(\gamma)$ is given by (A.5). Using (A.2) and (A.3) we write

$$\frac{\frac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}} = -b_0\lambda_1^{(1)}\sqrt{\frac{b_1\lambda_0^{(0)}}{\gamma}}\;\frac{\{\gamma[2(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + 2b_0\lambda_1^{(1)}\}^{-1/2}}{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}}. \tag{A.20}$$

Combining (A.19) and (A.20) we obtain

$$\frac{\eta^3\frac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}\,\frac{\partial\eta^2}{\partial\gamma}} = -\frac{k\lambda_1^{(1)}}{\psi(\gamma)}\, b_0^{3/2} b_1^{1/2}$$
$$\times\left[\frac{\{\gamma(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}+b_1\lambda_0^{(0)}) + b_0\lambda_1^{(1)}\}\{\gamma[p(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + pb_0\lambda_1^{(1)}\}}{\gamma[2(b_1\lambda_1^{(0)}-b_0\lambda_1^{(1)}) + b_1\lambda_0^{(0)}] + 2b_0\lambda_1^{(1)}}\right]^{1/2} \tag{A.21}$$

Since $\lim_{\gamma\to 0}\psi(\gamma) = -p(b_0\lambda_1^{(1)})^2$ we have

$$\lim_{\gamma\to 0}\frac{\eta^3\frac{\partial\rho}{\partial\gamma}}{\sqrt{1-\rho^2}\,\frac{\partial\eta^2}{\partial\gamma}} = k\sqrt{\frac{b_1}{2\lambda_1^{(1)}p}}. \tag{A.22}$$

Substituting (A.22) in (A.18) we obtain the desired result (5.6).

APPENDIX 3

Tables of exact (discrete) optimal designs

Tables A3.1-I and A3.1-II ($p = 2$, $k = 2$)

Tables A3.2-I and A3.2-II ($p = 3$, $k = 2$)

Tables A3.3-I and A3.3-II ($p = 4$, $k = 2$)

Tables A3.4-I and A3.4-II ($p = 5$, $k = 2$)

Tables A3.5-I and A3.5-II ($p = 6$, $k = 2$)

Tables A3.6-I and A3.6-II ($p = 3$, $k = 3$)

Table A3.1-I

Optimal Design[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 2$, $k = 2$, $b = 2(1)40$

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad b = 2f_0 + f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
2	1 0.3094	1 0.3738	1 0.4413	1 0.5101	1 0.5780	1 0.6431	1 0.7038	1 0.7587	1 0.8072	1 0.8489
3	1 0.4048	1 0.4795	1 0.5547	1 0.6278	1 0.6965	1 0.7586	1 0.8130	1 0.8590	1 0.8965	1 0.9262
4	1 0.4409	1 0.5178	1 0.5939	1 0.6666	1 0.7334	1 0.7926	2 0.8450	2 0.8934	2 0.9294	2 0.9550
5	1 0.4613	1 0.5390	1 0.6151	2 0.7020	2 0.7844	2 0.8515	2 0.9026	2 0.9393	2 0.9641	2 0.9798
6	1 0.4747	1 0.5528	2 0.6450	2 0.7381	2 0.8166	2 0.8782	2 0.9235	2 0.9546	2 0.9745	2 0.9865
7	1 0.4845	2 0.5650	2 0.6679	2 0.7592	3 0.8440	3 0.9058	3 0.9471	3 0.9723	3 0.9865	3 0.9939
8	1 0.4919	2 0.5810	3 0.6872	3 0.7902	3 0.8695	3 0.9250	3 0.9602	3 0.9805	3 0.9912	3 0.9963
9	1 0.4979	2 0.5930	3 0.7092	3 0.8094	4 0.8860	4 0.9394	4 0.9706	4 0.9870	4 0.9947	4 0.9981
10	1 0.5028	3 0.6056	3 0.7248	4 0.8303	4 0.9057	4 0.9526	4 0.9785	4 0.9912	4 0.9968	4 0.9989
11	1 0.5069	3 0.6182	4 0.7437	4 0.8471	4 0.9179	5 0.9605	5 0.9834	5 0.9938	5 0.9979	5 0.9994
12	1 0.5104	3 0.6282	4 0.7586	5 0.8619	5 0.9311	5 0.9696	5 0.9882	5 0.9959	5 0.9988	5 0.9997
13	1 0.5134	4 0.6406	5 0.7733	5 0.8764	5 0.9407	5 0.9751	5 0.9908	5 0.9970	5 0.9992	5 0.9998
14	2 0.5170	4 0.6509	5 0.7872	6 0.8871	6 0.9493	6 0.9803	6 0.9934	6 0.9981	6 0.9995	6 0.9999
15	2 0.5210	5 0.6609	6 0.7988	6 0.8995	6 0.9568	6 0.9841	6 0.9950	6 0.9986	6 0.9997	6 0.9999
16	2 0.5245	5 0.6712	6 0.8117	6 0.9085	7 0.9625	7 0.9871	7 0.9962	7 0.9991	7 0.9998	7 1.0000
17	3 0.5279	5 0.6798	6 0.8218	7 0.9179	7 0.9683	7 0.9897	7 0.9972	7 0.9994	7 0.9999	
18	3 0.5319	6 0.6898	7 0.8329	7 0.9256	7 0.9724	8 0.9915	8 0.9978	8 0.9995	8 0.9999	
19	3 0.5356	6 0.6983	7 0.8423	8 0.9327	8 0.9766	8 0.9933	8 0.9984	8 0.9997	8 1.0000	

[1/] The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.1-I (continued)

b	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
20	1 0.4877	4 0.5390	6 0.6209	7 0.7069	7 0.7855	8 0.8515	8 0.9026	8 0.9398	8 0.9641	8 0.9798
21	1 0.4890	4 0.5429	6 0.6276	7 0.7153	8 0.7946	8 0.8601	8 0.9095	9 0.9447	9 0.9682	9 0.9827
22	1 0.4903	4 0.5465	6 0.6335	8 0.7229	8 0.8032	9 0.8677	9 0.9166	9 0.9503	9 0.9720	9 0.9851
23	1 0.4914	5 0.5499	7 0.6399	8 0.7310	9 0.8110	9 0.8757	9 0.9227	9 0.9547	10 0.9750	10 0.9871
24	1 0.4925	5 0.5537	7 0.6459	8 0.7381	9 0.8191	9 0.8822	10 0.9285	10 0.9592	10 0.9781	10 0.9890
25	1 0.4935	5 0.5572	8 0.6517	9 0.7456	9 0.8260	10 0.8893	10 0.9339	10 0.9629	10 0.9805	10 0.9904
26	1 0.4945	6 0.5606	8 0.6576	9 0.7525	10 0.8335	10 0.8953	11 0.9385	11 0.9664	11 0.9829	11 0.9918
27	1 0.4954	6 0.5642	8 0.6630	10 0.7592	10 0.8401	11 0.9013	11 0.9433	11 0.9696	11 0.9848	11 0.9929
28	1 0.4962	6 0.5676	9 0.6688	10 0.7660	11 0.8467	11 0.9068	11 0.9472	12 0.9723	12 0.9865	12 0.9939
29	1 0.4970	7 0.5710	9 0.6741	11 0.7720	11 0.8528	12 0.9119	12 0.9513	12 0.9750	12 0.9881	12 0.9947
30	1 0.4978	7 0.5744	10 0.6794	11 0.7785	12 0.8586	12 0.9169	12 0.9548	12 0.9772	13 0.9894	13 0.9954
31	1 0.4985	7 0.5776	10 0.6847	11 0.7842	12 0.8645	12 0.9213	13 0.9581	13 0.9794	13 0.9907	13 0.9961
32	1 0.4992	8 0.5810	10 0.6896	12 0.7902	13 0.8696	13 0.9259	13 0.9612	13 0.9813	13 0.9917	13 0.9966
33	1 0.4998	8 0.5843	11 0.6948	12 0.7958	13 0.8750	13 0.9298	14 0.9639	14 0.9830	14 0.9926	14 0.9971
34	1 0.5004	9 0.5874	11 0.6996	13 0.8011	13 0.8798	14 0.9338	14 0.9666	14 0.9846	14 0.9935	14 0.9975
35	1 0.5010	9 0.5908	12 0.7045	13 0.8066	14 0.8847	14 0.9374	14 0.9689	15 0.9859	15 0.9942	15 0.9978
36	1 0.5016	9 0.5939	12 0.7092	14 0.8114	14 0.8892	15 0.9408	15 0.9713	15 0.9873	15 0.9949	15 0.9981
37	1 0.5021	10 0.5970	13 0.7137	14 0.8167	15 0.8936	15 0.9441	15 0.9733	15 0.9884	16 0.9954	16 0.9984
38	2 0.5027	10 0.6002	13 0.7184	14 0.8214	15 0.8978	16 0.9470	16 0.9752	16 0.9895	16 0.9960	16 0.9986
39	2 0.5034	10 0.6032	13 0.7227	15 0.8262	16 0.9017	16 0.9500	16 0.9770	16 0.9904	16 0.9964	16 0.9988
40	2 0.5041	11 0.6063	14 0.7272	15 0.8307	16 0.9057	16 0.9526	17 0.9786	17 0.9913	17 0.9968	17 0.9989

Table A3.1-II

Optimal Design[1/] to Achieve a Specified Confidence Coefficient as a Function of d/σ for $p = 2, k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 \\ 1 & 2 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad b = 2f_0 + f_1$$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	616 258	154 41	69 29	39 16	25 10	18 8	13 5	10 4	8 3	7 3
0.95	351 144	88 36	39 16	22 9	15 6	10 4	8 3	6 2	5 2	4 2
0.90	241 96	61 24	27 11	16 6	10 4	7 3	5 2	5 2	4 2	3 1
0.85	179 69	45 17	20 8	12 5	8 3	5 2	5 2	3 1	3 1	3 1
0.80	136 51	34 13	16 6	9 3	6 2	5 2	3 1	3 1	2 1	2 1

[1/] The upper entry in each cell is b, and the lower entry is $\hat{f}_0$.

Table A3.2-I

Optimal Design[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 3$, $k = 2$, $b = 3(3)114$

$$D_0 = \left\{\begin{matrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}\right\}, \quad b = 3f_0 + 3f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
3	1 0.1721	1 0.2285	1 0.2932	1 0.3643	1 0.4394	1 0.5157	1 0.5904	1 0.6609	1 0.7252	1 0.7821
6	1 0.3279	1 0.4138	1 0.5038	1 0.5932	1 0.6778	1 0.7537	1 0.8186	1 0.8714	1 0.9123	1 0.9425
9	1 0.3852	1 0.4770	1 0.5700	2 0.6625	2 0.7650	2 0.8468	2 0.9066	2 0.9468	2 0.9717	2 0.9859
12	1 0.4170	1 0.5111	2 0.6144	2 0.7288	3 0.8270	3 0.9023	3 0.9497	3 0.9763	3 0.9898	3 0.9960
15	1 0.4380	1 0.5330	2 0.6537	3 0.7806	3 0.8748	4 0.9369	4 0.9722	4 0.9891	4 0.9961	4 0.9988
18	1 0.4531	2 0.5542	3 0.6935	4 0.8212	4 0.9101	4 0.9605	4 0.9848	4 0.9949	4 0.9985	4 0.9996
21	1 0.4646	3 0.5741	4 0.7273	5 0.8537	5 0.9347	5 0.9751	5 0.9919	5 0.9977	5 0.9995	5 0.9999
24	1 0.4738	3 0.5954	5 0.7568	5 0.8804	6 0.9521	6 0.9841	6 0.9956	6 0.9990	6 0.9998	6 1.0000
27	1 0.4813	4 0.6154	6 0.7828	6 0.9030	7 0.9647	7 0.9897	7 0.9976	7 0.9995	7 0.9999	
30	1 0.4876	5 0.6340	6 0.8067	7 0.9210	7 0.9744	7 0.9934	7 0.9986	7 0.9998	7 1.0000	
33	1 0.4929	5 0.6519	7 0.8280	8 0.9354	8 0.9813	8 0.9958	8 0.9993	8 0.9999		
36	1 0.4976	6 0.6696	8 0.8468	9 0.9470	9 0.9863	9 0.9973	9 0.9996	9 1.0000		
39	1 0.5017	7 0.6861	9 0.8633	9 0.9567	10 0.9899	10 0.9982	10 0.9998			
42	1 0.5053	8 0.7016	10 0.8779	10 0.9647	10 0.9926	10 0.9989	10 0.9999			
45	2 0.5092	8 0.7163	10 0.8913	11 0.9711	11 0.9946	11 0.9993	11 0.9999			
48	2 0.5136	9 0.7307	11 0.9032	12 0.9763	12 0.9960	12 0.9995	12 1.0000			

1/ The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.2-I (continued)

b	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
51	1 0.4657	3 0.5184	8 0.6261	10 0.7443	11 0.8433	12 0.9136	12 0.9570	12 0.9806	13 0.9920	13 0.9970
54	1 0.4681	4 0.5231	9 0.6364	11 0.7570	12 0.8554	13 0.9229	13 0.9632	13 0.9841	13 0.9938	13 0.9978
57	1 0.4703	4 0.5283	9 0.6468	12 0.7690	13 0.8664	13 0.9311	14 0.9684	14 0.9870	14 0.9952	14 0.9984
60	1 0.4723	5 0.5334	10 0.6568	12 0.7806	14 0.8766	14 0.9386	15 0.9728	15 0.9893	15 0.9962	15 0.9988
63	1 0.4742	5 0.5385	11 0.6664	13 0.7917	14 0.8861	15 0.9452	15 0.9767	16 0.9912	16 0.9971	16 0.9991
66	1 0.4760	6 0.5438	11 0.6758	14 0.8021	15 0.8948	16 0.9511	16 0.9800	16 0.9928	16 0.9977	16 0.9993
69	1 0.4776	7 0.5490	12 0.6851	15 0.8119	16 0.9029	17 0.9563	17 0.9828	17 0.9941	17 0.9982	17 0.9995
72	1 0.4791	7 0.5542	13 0.6940	15 0.8212	17 0.9103	17 0.9610	18 0.9853	18 0.9951	18 0.9986	18 0.9996
75	1 0.4806	8 0.5596	14 0.7027	16 0.8303	18 0.9170	18 0.9652	18 0.9873	19 0.9960	19 0.9989	19 0.9997
78	1 0.4819	9 0.5648	15 0.7110	17 0.8388	18 0.9234	19 0.9689	19 0.9891	19 0.9967	19 0.9991	19 0.9998
81	1 0.4832	9 0.5700	15 0.7193	18 0.8468	19 0.9293	20 0.9722	20 0.9906	20 0.9973	20 0.9993	20 0.9999
84	1 0.4844	10 0.5752	16 0.7273	19 0.8544	20 0.9347	20 0.9751	21 0.9920	21 0.9978	21 0.9995	21 0.9999
87	1 0.4855	11 0.5804	17 0.7351	19 0.8616	21 0.9396	21 0.9778	22 0.9931	22 0.9982	22 0.9996	22 0.9999
90	1 0.4866	12 0.5855	18 0.7426	20 0.8686	21 0.9442	22 0.9802	22 0.9940	22 0.9985	22 0.9997	22 0.9999
93	1 0.4876	12 0.5906	18 0.7499	21 0.8751	22 0.9484	23 0.9822	23 0.9949	23 0.9988	23 0.9997	23
96	1 0.4886	13 0.5957	19 0.7570	22 0.8813	23 0.9523	24 0.9841	24 0.9956	24 0.9990	24 0.9998	
99	1 0.4896	14 0.6006	20 0.7640	23 0.8872	24 0.9559	24 0.9858	25 0.9962	25 0.9992	25 0.9998	
102	1 0.4904	14 0.6056	21 0.7706	23 0.8928	25 0.9592	25 0.9873	25 0.9967	25 0.9993	25 0.9999	
105	1 0.4913	15 0.6105	22 0.7771	24 0.8981	25 0.9623	26 0.9886	26 0.9972	26 0.9994	26 0.9999	
108	1 0.4921	16 0.6154	22 0.7835	25 0.9032	26 0.9652	27 0.9898	27 0.9976	27 0.9995	27 0.9999	
111	1 0.4929	17 0.6201	23 0.7897	26 0.9080	27 0.9678	27 0.9909	28 0.9979	28 0.9996	28 0.9999	
114	1 0.4936	17 0.6249	24 0.7957	27 0.9125	28 0.9702	28 0.9919	28 0.9982	28 0.9997	28 1.0000	

Table A3.2-II

Optimal Design[1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 3,\ k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad b = 3f_0 + 3f_1$$

Confidence Coefficient ($1-\alpha$)	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	975 241	246 61	111 27	63 16	42 10	30 7	21 5	18 4	15 4	12 3
0.95	591 141	150 36	66 16	39 9	24 6	18 4	15 4	12 3	9 2	9 2
0.90	426 98	108 25	48 11	27 6	18 4	12 3	9 2	9 2	6 1	6 1
0.85	330 73	84 19	39 9	21 5	15 3	12 3	9 2	6 1	6 1	6 1
0.80	264 56	66 14	30 6	18 4	12 3	9 2	6 1	6 1	6 1	6 1

[1/] The upper entry in each cell is b, and the lower entry is $\hat{f}_0$.

Table A3.3-I

Optimal Designs[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 4,\ k = 2,\ b = 4,8(2)78$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{Bmatrix},\quad b = 4f_0 + 6f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
4	1 0.0957	1 0.1397	1 0.1948	1 0.2602	1 0.3341	1 0.4136	1 0.4953	1 0.5757	1 0.6516	1 0.7206
8	2 0.1126	2 0.1845	2 0.2774	2 0.3859	2 0.5011	2 0.6132	2 0.7140	2 0.7982	2 0.8638	2 0.9121
10	1 0.2812	1 0.3752	1 0.4769	1 0.5797	1 0.6769	1 0.7630	1 0.8343	1 0.8898	1 0.9303	1 0.9582
12	3 0.1268	3 0.2239	3 0.3493	3 0.4894	3 0.6265	3 0.7454	3 0.8381	3 0.9036	3 0.9462	3 0.9717
14	2 0.2483	2 0.3724	2 0.5101	2 0.6452	2 0.7629	2 0.8542	2 0.9176	2 0.9572	2 0.9796	2 0.9910
16	1 0.3519	1 0.4559	1 0.5628	1 0.6651	1 0.7562	4 0.8323	4 0.9079	4 0.9535	4 0.9784	4 0.9907
18	3 0.2374	3 0.3849	3 0.5490	3 0.7025	3 0.8245	3 0.9076	3 0.9565	3 0.9817	3 0.9931	3 0.9976
20	2 0.3188	2 0.4592	2 0.6041	2 0.7350	2 0.8388	2 0.9112	2 0.9559	2 0.9802	2 0.9920	2 0.9971
22	1 0.3913	1 0.4989	1 0.6064	4 0.7515	4 0.8697	4 0.9406	4 0.9763	4 0.9917	4 0.9975	4 0.9993
24	3 0.3039	3 0.4709	3 0.6415	3 0.7862	3 0.8885	3 0.9493	3 0.9800	3 0.9931	3 0.9979	3 0.9994
26	2 0.3631	2 0.5103	2 0.6556	5 0.7929	5 0.9030	5 9.9614	5 0.9868	5 0.9961	5 0.9990	5 0.9998
28	1 0.4172	1 0.5264	4 0.6758	4 0.8260	4 0.9211	4 0.9698	4 0.9902	4 0.9973	4 0.9994	4 0.9999
30	3 0.3490	3 0.5249	3 0.6948	3 0.8297	6 0.9277	6 0.9747	6 0.9926	6 0.9982	6 0.9996	6 0.9999
32	2 0.3941	2 0.5446	5 0.7071	5 0.8578	5 0.9434	5 0.9815	5 0.9950	5 0.9989	5 0.9998	5 1.0000
34	1 0.4358	1 0.5458	4 0.7283	4 0.8657	7 0.9459	7 0.9833	7 0.9958	7 0.9991	7 0.9998	
36	3 0.3819	3 0.5623	6 0.7356	6 0.8834	6 0.9590	6 0.9884	6 0.9974	6 0.9995	6 0.9999	

[1/] The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.3-I (continued)

b	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
38	2 0.3442	2 0.4172	2 0.4932	2 0.5694	5 0.6636	5 0.7574	5 0.8343	5 0.8929	5 0.9345	5 0.9622
40	1 0.3957	1 0.4500	1 0.5053	4 0.5797	4 0.6769	4 0.7630	7 0.8438	7 0.9041	7 0.9447	7 0.9700
42	3 0.3204	3 0.4072	3 0.4986	3 0.5899	6 0.6866	6 0.7830	6 0.8589	6 0.9139	6 0.9507	6 0.9735
44	2 0.3609	2 0.4353	2 0.5121	5 0.5968	5 0.7014	5 0.7911	8 0.8650	8 0.9210	8 0.9568	8 0.9779
46	1 0.4066	1 0.4613	1 0.5168	4 0.6089	7 0.7081	7 0.8057	7 0.8795	7 0.9304	7 0.9626	7 0.9812
48	3 0.3385	3 0.4274	3 0.5200	6 0.6134	6 0.7237	6 0.8151	6 0.8844	9 0.9348	9 0.9662	9 0.9837
50	2 0.3746	2 0.4500	2 0.5272	5 0.6267	5 0.7293	8 0.8259	8 0.8969	8 0.9435	8 0.9713	8 0.9865
52	1 0.4156	1 0.4706	4 0.5285	4 0.6317	7 0.7441	7 0.8360	7 0.9026	7 0.9465	10 0.9734	10 0.9879
54	3 0.3536	3 0.4440	3 0.5372	6 0.6436	6 0.7513	9 0.8439	9 0.9115	9 0.9539	9 0.9779	9 0.9902
56	2 0.3860	2 0.4621	2 0.5396	5 0.6503	8 0.7629	8 0.8542	8 0.9176	8 0.9572	8 0.9796	8 0.9910
58	1 0.4231	1 0.4783	4 0.5472	7 0.6597	7 0.7711	10 0.8599	10 0.9240	10 0.9623	10 0.9829	10 0.9929
60	3 0.3663	3 0.4579	3 0.5516	6 0.6675	9 0.7801	9 0.8701	9 0.9300	9 0.9656	9 0.9846	9 0.9937
62	2 0.3957	2 0.4724	5 0.5570	8 0.6750	8 0.7890	8 0.8746	11 0.9346	11 0.9691	11 0.9867	11 0.9948
64	1 0.4295	1 0.4849	4 0.5628	7 0.6837	10 0.7961	10 0.8842	10 0.9404	10 0.9722	10 0.9883	10 0.9955
66	3 0.3773	3 0.4697	6 0.5668	9 0.6896	9 0.8053	9 0.8892	12 0.9437	12 0.9746	12 0.9896	12 0.9961
68	2 0.4041	2 0.4813	5 0.5736	8 0.6989	11 0.8109	11 0.8966	11 0.9492	11 0.9775	11 0.9910	11 0.9967
70	1 0.4351	1 0.4907	7 0.5766	10 0.7036	10 0.8202	10 0.9018	10 0.9518	13 0.9791	13 0.9919	13 0.9971
72	3 0.3869	3 0.4800	6 0.5840	9 0.7133	9 0.8249	12 0.9076	12 0.9565	12 0.9817	12 0.9931	12 0.9976
74	2 0.4115	2 0.4890	5 0.5877	8 0.7186	11 0.8338	11 0.9129	11 0.9592	11 0.9829	14 0.9936	14 0.9979
76	1 0.4400	1 0.4957	7 0.5942	10 0.7269	10 0.8390	13 0.9173	13 0.9627	13 0.9850	13 0.9946	13 0.9983
78	3 0.3953	3 0.4890	6 0.5987	9 0.7328	12 0.8463	12 0.9226	12 0.9654	12 0.9863	12 0.9951	12 0.9985

Table A3.3-II

Optimal Design[1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 4$, $k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{Bmatrix}, \quad b = 4f_0 + 6f_1$$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	1342 226	336 57	150 24	86 14	54 9	40 7	28 4	22 4	18 3	14 2
0.95	838 136	210 33	94 16	54 9	36 6	26 5	18 3	14 2	14 2	10 1
0.90	622 97	156 24	70 10	40 7	26 5	18 3	14 2	12 3	10 1	8 2
0.85	496 73	124 19	56 8	32 5	22 4	14 2	14 2	10 1	8 2	8 2
0.80	404 56	102 15	46 7	28 4	18 3	14 2	10 1	10 1	8 2	8 2

[1/] The upper entry in each cell is b, and the lower entry is $\dot{f}_0$.

Table 3.4-I

Optimal Design[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 5$, $k = 2$, $b = 5(5)200$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 & 5 & 3 & 4 & 5 & 4 & 5 & 5 \end{Bmatrix}, \quad b = 5f_0 + 10f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
5	1 0.0532	1 0.0854	1 0.1294	1 0.1858	1 0.2540	1 0.3316	1 0.4155	1 0.5014	1 0.5854	1 0.6639
10	2 0.0652	2 0.1209	2 0.2013	2 0.3041	2 0.4216	2 0.5427	2 0.6564	2 0.7544	2 0.8328	2 0.8913
15	1 0.2498	1 0.3504	1 0.4623	1 0.5768	1 0.6846	1 0.7781	1 0.8530	1 0.9085	1 0.9465	1 0.9706
20	2 0.2097	2 0.3393	2 0.4898	2 0.6400	2 0.7696	2 0.8667	2 0.9304	2 0.9672	2 0.9860	2 0.9946
25	1 0.3297	1 0.4443	1 0.5636	3 0.6962	3 0.8298	3 0.9167	3 0.9643	3 0.9866	3 0.9955	3 0.9987
30	2 0.2896	2 0.4427	2 0.6039	2 0.7480	4 0.8738	4 0.9469	4 0.9809	4 0.9941	4 0.9984	4 0.9996
35	1 0.3746	1 0.4941	3 0.6405	3 0.7986	5 0.9061	5 0.9657	5 0.9895	5 0.9973	5 0.9994	5 0.9999
40	2 0.3404	2 0.5031	4 0.6740	4 0.8373	4 0.9338	4 0.9780	6 0.9941	6 0.9987	6 0.9998	6 1.0000
45	1 0.4041	1 0.5258	5 0.7047	5 0.8677	5 0.9533	5 0.9869	5 0.9971	5 0.9995	5 0.9999	
50	2 0.3760	2 0.5432	4 0.7375	6 0.8920	6 0.9666	6 0.9920	6 0.9985	6 0.9998	6 1.0000	
55	1 0.4254	3 0.5602	5 0.7662	7 0.9115	7 0.9758	7 0.9950	7 0.9992	7 0.9999		
60	2 0.4026	4 0.5768	6 0.7913	8 0.9273	8 0.9824	8 0.9968	8 0.9996	8 1.0000		
65	1 0.4416	5 0.5932	7 0.8135	7 0.9415	9 0.9870	9 0.9980	9 0.9998			
70	2 0.4234	4 0.6115	8 0.8331	8 0.9529	8 0.9908	8 0.9987	8 0.9999			
75	1 0.4544	5 0.6291	9 0.8505	9 0.9619	9 0.9934	9 0.9992	9 0.9999			
80	2 0.4402	6 0.6457	8 0.8667	10 0.9691	10 0.9953	10 0.9995	10 1.0000			
85	1 0.4650	7 0.6615	9 0.8818	11 0.9748	11 0.9966	11 0.9997				

[1/] The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.4-I (continued)

b	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
90	2 0.3696	2 0.4541	4 0.5453	8 0.6766	10 0.8021	10 0.8950	10 0.9505	12 0.9794	12 0.9924	12 0.9975
95	1 0.4122	1 0.4738	5 0.5547	9 0.6910	9 0.8167	11 0.9065	11 0.9580	11 0.9833	11 0.9941	11 0.9982
100	2 0.3807	2 0.4659	6 0.5640	8 0.7054	10 0.8311	12 0.9167	12 0.9643	12 0.9866	12 0.9955	12 0.9987
105	1 0.4195	1 0.4813	5 0.5741	9 0.7197	11 0.8442	13 0.9257	13 0.9696	13 0.9891	13 0.9966	13 0.9991
110	2 0.3902	2 0.4760	6 0.5843	10 0.7331	12 0.8562	12 0.9337	14 0.9740	14 0.9911	14 0.9974	14 0.9993
115	1 0.4258	1 0.4878	7 0.5942	11 0.7458	13 0.8672	13 0.9412	15 0.9778	15 0.9928	15 0.9980	15 0.9995
120	2 0.3986	2 0.4848	8 0.6039	12 0.7578	14 0.8773	14 0.9479	14 0.9811	16 0.9941	16 0.9984	16 0.9996
125	1 0.4314	1 0.4935	9 0.6134	13 0.7693	15 0.8866	15 0.9537	15 0.9839	15 0.9952	15 0.9988	15 0.9997
130	2 0.4060	2 0.4926	10 0.6226	12 0.7801	14 0.8954	16 0.9588	16 0.9863	16 0.9961	16 0.9991	16 0.9998
135	1 0.4364	1 0.4986	9 0.6321	13 0.7911	15 0.9037	17 0.9633	17 0.9883	17 0.9969	17 0.9993	17 0.9999
140	2 0.4126	2 0.4995	10 0.6415	14 0.8014	16 0.9112	18 0.9673	18 0.9900	18 0.9974	18 0.9994	18 0.9999
145	1 0.4408	1 0.5031	11 0.6507	15 0.8112	17 0.9181	17 0.9710	19 0.9915	19 0.9979	19 0.9996	19 0.9999
150	2 0.4186	2 0.5057	12 0.6596	16 0.8205	18 0.9245	18 0.9742	18 0.9927	18 0.9983	18 0.9997	18 0.9999
155	1 0.4448	3 0.5085	13 0.6683	17 0.8292	19 0.9303	19 0.9771	19 0.9938	19 0.9986	19 0.9997	19 1.0000
160	2 0.4240	4 0.5115	14 0.6767	18 0.8375	18 0.9357	20 0.9796	20 0.9947	20 0.9989	20 0.9998	
165	1 0.4484	3 0.5152	13 0.6851	17 0.8456	19 0.9408	21 0.9819	21 0.9955	21 0.9991	21 0.9998	
170	2 0.4289	4 0.5190	14 0.6935	18 0.8534	20 0.9455	22 0.9838	22 0.9962	22 0.9993	22 0.9999	
175	1 0.4518	5 0.5228	15 0.7017	19 0.8607	21 0.9497	21 0.9857	21 0.9967	23 0.9994	23 0.9999	
180	2 0.4334	6 0.5267	16 0.7096	20 0.8677	22 0.9537	22 0.9873	22 0.9972	22 0.9995	22 0.9999	
185	1 0.4548	7 0.5306	17 0.7173	21 0.8743	23 0.9572	23 0.9887	23 0.9976	23 0.9996	23 0.9999	
190	2 0.4375	6 0.5346	18 0.7248	22 0.8805	22 0.9605	24 0.9899	24 0.9980	24 0.9997	24 1.0000	
195	1 0.4576	7 0.5389	19 0.7321	23 0.8864	23 0.9637	25 0.9910	25 0.9983	25 0.9997		
200	2 0.4414	8 0.5432	18 0.7394	22 0.8922	24 0.9666	24 0.9920	26 0.9985	26 0.9998		

Table A3.4-II

Optimal Design [1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 5,\ k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 & 5 & 3 & 4 & 5 & 4 & 5 & 5 \end{Bmatrix},\quad b = 5f_0 + 10f_1$$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	1715 213	430 54	195 25	110 14	70 8	50 6	40 6	30 4	25 3	20 2
0.95	1100 132	275 33	125 15	70 8	45 5	35 5	25 3	20 2	20 2	15 1
0.90	830 94	210 24	95 11	55 7	35 5	25 3	20 2	15 1	15 1	15 1
0.85	675 73	170 18	75 9	45 5	30 4	20 2	15 1	15 1	15 1	10 2
0.80	560 58	140 14	65 7	40 4	25 3	20 2	15 1	15 1	10 2	10 2

[1/] The upper entry in each cell is b, and the lower entry is $\hat{f}_0$.

Table A3.5-I

Optimal Design[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 6,\ k = 2,\ b = 6(6)18(3)123$

$$D_0 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\},\quad D_1 = \left\{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 5 & 6 & 6 \end{matrix}\right\}$$

$$b = 6f_0 + 15f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
6	1 0.0296	1 0.0522	1 0.0860	1 0.1327	1 0.1931	1 0.2660	1 0.3485	1 0.4368	1 0.5260	1 0.6117
12	2 0.0378	2 0.0793	2 0.1461	2 0.2397	2 0.3547	2 0.4802	2 0.6034	2 0.7131	2 0.8029	2 0.8710
18	3 0.0452	3 0.1060	3 0.2065	3 0.3424	3 0.4959	3 0.6436	3 0.7672	3 0.8590	3 0.9203	3 0.9578
21	1 0.2272	1 0.3337	1 0.4549	1 0.5799	1 0.6966	1 0.7956	1 0.8718	1 0.9254	1 0.9597	1 0.9798
24	4 0.0522	4 0.1327	4 0.2660	4 0.4368	4 0.6117	4 0.7594	4 0.8651	4 0.9311	4 0.9677	4 0.9860
27	2 0.1824	2 0.3165	2 0.4785	2 0.6421	2 0.7810	2 0.8810	2 0.9427	2 0.9755	2 0.9907	2 0.9968
30	5 0.0591	5 0.1595	5 0.3237	5 0.5211	5 0.7035	5 0.8387	5 0.9221	5 0.9662	5 0.9868	5 0.9953
33	3 0.1642	3 0.3195	3 0.5117	3 0.6972	3 0.8393	3 0.9271	3 0.9716	3 0.9905	3 0.9972	3 0.9993
36	1 0.3140	1 0.4382	1 0.5686	1 0.6916	1 0.7958	6 0.8922	6 0.9549	6 0.9834	6 0.9945	6 0.9984
39	4 0.1560	4 0.3305	4 0.5473	4 0.7452	4 0.8814	4 0.9542	4 0.9852	4 0.9959	4 0.9990	4 0.9998
42	2 0.2689	2 0.4335	2 0.6094	2 0.7640	2 0.8764	2 0.9441	2 0.9783	2 0.9927	2 0.9979	2 0.9995
45	5 0.1527	5 0.3456	5 0.5829	5 0.7862	5 0.9120	5 0.9707	5 0.9920	5 0.9982	5 0.9997	5 0.9999
48	3 0.2462	3 0.4392	3 0.6457	3 0.8140	3 0.9197	3 0.9716	3 0.9917	3 0.9980	3 0.9996	3 0.9999
51	1 0.3631	1 0.4935	1 0.6245	6 0.8210	6 0.9345	6 0.9810	6 0.9955	6 0.9991	6 0.9999	6 1.0000
54	4 0.2333	4 0.4498	4 0.6788	4 0.8513	4 0.9459	4 0.9845	4 0.9965	4 0.9993	4 0.9999	
57	2 0.3247	2 0.5016	2 0.6773	7 0.8504	7 0.9512	7 0.9876	7 0.9975	7 0.9996	7 0.9999	

[1/] The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.5-I (continued)

b	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
60	5 0.1384	5 0.2259	5 0.3370	5 0.4631	5 0.5914	5 0.7090	5 0.8066	5 0.8801	5 0.9307	5 0.9626
63	3 0.2146	3 0.3033	3 0.4051	3 0.5132	3 0.6199	3 0.7177	3 0.8010	8 0.8750	8 0.9300	8 0.9634
66	1 0.3323	1 0.3955	1 0.4615	1 0.5285	6 0.6147	6 0.7366	6 0.8335	6 0.9027	6 0.9474	6 0.9736
69	4 0.1934	4 0.2902	4 0.4044	4 0.5267	4 0.6456	4 0.7510	4 0.8363	4 0.8994	9 0.9445	9 0.9726
72	2 0.2806	2 0.3639	2 0.4540	2 0.5464	7 0.6374	7 0.7617	7 0.8565	7 0.9207	7 0.9597	7 0.9811
75	5 0.1785	5 0.2820	5 0.4070	5 0.5411	5 0.6694	5 0.7793	5 0.8638	5 0.9224	5 0.9592	5 0.9802
78	3 0.2490	3 0.3454	3 0.4528	3 0.5634	3 0.6688	8 0.7846	8 0.8762	8 0.9352	8 0.9690	8 0.9864
81	1 0.3541	1 0.4188	1 0.4857	6 0.5560	6 0.6916	6 0.8038	6 0.8859	6 0.9394	6 0.9706	6 0.9869
84	4 0.2272	4 0.3337	4 0.4549	4 0.5799	4 0.6966	9 0.8053	9 0.8931	9 0.9468	9 0.9760	9 0.9901
87	2 0.3065	2 0.3932	2 0.4855	2 0.5784	7 0.7124	7 0.8253	7 0.9039	7 0.9522	7 0.9784	7 0.9911
90	5 0.2114	5 0.3260	5 0.4591	5 0.5960	5 0.7211	10 0.8242	10 0.9076	10 0.9562	10 0.9813	10 0.9927
93	3 0.2763	3 0.3779	3 0.4885	3 0.5997	8 0.7319	8 0.8441	8 0.9187	8 0.9619	8 0.9839	8 0.9939
96	1 0.3708	1 0.4365	1 0.5040	6 0.6118	6 0.7431	6 0.8455	11 0.9200	11 0.9639	11 0.9854	11 0.9946
99	4 0.2548	4 0.3679	4 0.4934	4 0.6189	9 0.7501	9 0.8607	9 0.9310	9 0.9695	9 0.9879	9 0.9957
102	2 0.3270	2 0.4161	2 0.5097	7 0.6271	7 0.7629	7 0.8647	7 0.9308	12 0.9702	12 0.9885	12 0.9960
105	5 0.2387	5 0.3613	5 0.4994	5 0.6367	10 0.7671	10 0.8754	10 0.9412	10 0.9754	10 0.9908	10 0.9969
108	3 0.2985	3 0.4038	3 0.5163	8 0.6421	8 0.7810	8 0.8810	8 0.9427	8 0.9755	13 0.9910	13 0.9970
111	1 0.3842	1 0.4506	1 0.5184	6 0.6534	11 0.7830	11 0.8885	11 0.9498	11 0.9801	11 0.9930	11 0.9978
114	4 0.2778	4 0.3957	4 0.5237	9 0.6565	9 0.7975	9 0.8951	9 0.9523	9 0.9809	9 0.9932	9 0.9979
117	2 0.3439	2 0.4347	2 0.5289	7 0.6692	7 0.7986	12 0.9001	12 0.9570	12 0.9838	12 0.9946	12 0.9984
120	5 0.2619	5 0.3904	5 0.5314	10 0.6706	10 0.8126	10 0.9073	10 0.9600	10 0.9849	10 0.9950	10 0.9985
123	3 0.3171	3 0.4250	3 0.5387	8 0.6841	8 0.8158	13 0.9105	13 0.9632	13 0.9868	13 0.9958	13 0.9988

Table A3.5-II

Optimal Design [1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 6,\ k = 2$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix},\quad D_1 = \begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 5 & 6 & 6 \end{Bmatrix}$$

$$b = 6f_0 + 15f_1$$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	2085 205	522 52	234 24	132 12	84 9	60 5	45 5	33 3	27 2	27 2
0.95	1359 129	342 32	153 13	87 7	57 7	39 4	33 3	27 2	21 1	18 3
0.90	1044 94	264 24	117 12	66 6	45 5	33 3	27 2	21 1	18 3	18 3
0.85	858 73	216 16	99 9	54 4	39 4	27 2	21 1	18 3	18 3	12 2
0.80	720 55	180 15	81 6	48 3	33 3	27 2	21 1	18 3	12 2	12 2

[1/] The upper entry in each cell is b, and the lower entry is $\hat{f}_0$.

Table A3.6-I

Optimal Design[1/] and Associated Confidence Coefficient $\hat{g}_E$

as a Function of d/σ

for $p = 3$, $k = 3$, $b = 3(1)40$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}, \quad b = 3f_0 + f_1$$

b	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
3	1 0.2864	1 0.3786	1 0.4776	1 0.5773	1 0.6716	1 0.7555	1 0.8258	1 0.8813	1 0.9226	1 0.9518
4	1 0.3407	1 0.4414	1 0.5456	1 0.6464	1 0.7374	1 0.8144	1 0.8753	1 0.9204	1 0.9518	1 0.9724
5	1 0.3750	1 0.4797	1 0.5855	1 0.6851	1 0.7726	1 0.8442	1 0.8990	1 0.9381	1 0.9641	1 0.9804
6	1 0.3991	1 0.5059	1 0.6121	2 0.7282	2 0.8302	2 0.9030	2 0.9494	2 0.9758	2 0.9895	2 0.9958
7	1 0.4172	1 0.5253	2 0.6411	2 0.7648	2 0.8598	2 0.9242	2 0.9629	2 0.9836	2 0.9934	2 0.9976
8	1 0.4314	1 0.5403	2 0.6696	2 0.7897	2 0.8790	2 0.9372	2 0.9707	2 0.9877	2 0.9954	2 0.9984
9	1 0.4430	1 0.5523	2 0.6909	3 0.8212	3 0.9101	3 0.9605	3 0.9848	3 0.9949	3 0.9985	3 0.9996
10	1 0.4526	2 0.5675	3 0.7162	3 0.8430	3 0.9247	3 0.9687	3 0.9888	3 0.9965	3 0.9991	3 0.9998
11	1 0.4608	2 0.5821	3 0.7371	3 0.8591	3 0.9350	3 0.9742	3 0.9912	3 0.9974	3 0.9994	3 0.9999
12	1 0.4678	2 0.5942	4 0.7555	4 0.8813	4 0.9518	4 0.9837	4 0.9954	4 0.9989	4 0.9998	4 1.0000
13	1 0.4740	3 0.6086	4 0.7750	4 0.8949	4 0.9593	4 0.9870	4 0.9966	4 0.9992	4 0.9999	
14	1 0.4794	3 0.6216	4 0.7907	4 0.9054	4 0.9649	4 0.9893	4 0.9973	4 0.9995	4 0.9999	
15	1 0.4843	4 0.6331	5 0.8068	5 0.9207	5 0.9740	5 0.9932	5 0.9986	5 0.9998	5 1.0000	
16	1 0.4886	4 0.6464	5 0.8212	5 0.9294	5 0.9779	5 0.9946	5 0.9989	5 0.9998		
17	1 0.4926	4 0.6579	5 0.8332	5 0.9363	5 0.9809	5 0.9955	5 0.9992	5 0.9999		
18	1 0.4962	5 0.6691	6 0.8468	6 0.9468	6 0.9859	6 0.9971	6 0.9996	6 0.9999		

[1/] The upper entry in each cell is $\hat{f}_0$ and the lower entry is $\hat{g}_E$.

Table A3.6-I (continued)

b	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
19	1 0.4439	1 0.4995	4 0.5771	5 0.6807	6 0.7783	6 0.8577	6 0.9148	6 0.9524	6 0.9753	6 0.9880
20	1 0.4469	1 0.5025	4 0.5855	5 0.6909	6 0.7897	6 0.8669	6 0.9216	6 0.9570	6 0.9781	6 0.9896
21	1 0.4496	1 0.5053	4 0.5930	6 0.7016	7 0.8004	7 0.8782	7 0.9313	7 0.9642	7 0.9827	7 0.9923
22	1 0.4522	1 0.5079	5 0.6002	6 0.7117	7 0.8112	7 0.8866	7 0.9372	7 0.9679	7 0.9849	7 0.9934
23	1 0.4546	1 0.5103	5 0.6081	7 0.7209	7 0.8206	7 0.8937	7 0.9421	7 0.9710	7 0.9866	7 0.9943
24	1 0.4568	2 0.5136	5 0.6152	7 0.7308	8 0.8302	8 0.9030	8 0.9494	8 0.9758	8 0.9895	8 0.9958
25	1 0.4589	2 0.5168	6 0.6223	7 0.7396	8 0.8390	8 0.9094	8 0.9536	8 0.9783	8 0.9908	8 0.9964
26	1 0.4608	2 0.5198	6 0.6296	8 0.7484	8 0.8468	8 0.9150	8 0.9572	8 0.9804	8 0.9918	8 0.9969
27	1 0.4627	3 0.5231	6 0.6364	8 0.7570	9 0.8554	9 0.9226	9 0.9626	9 0.9837	9 0.9935	9 0.9977
28	1 0.4644	3 0.5266	7 0.6433	8 0.7648	9 0.8626	9 0.9276	9 0.9657	9 0.9853	9 0.9943	9 0.9980
29	1 0.4661	3 0.5299	7 0.6501	9 0.7731	9 0.8691	9 0.9320	9 0.9683	9 0.9867	9 0.9950	9 0.9983
30	1 0.4676	4 0.5332	7 0.6565	9 0.7806	10 0.8767	10 0.9382	10 0.9723	10 0.9889	10 0.9960	10 0.9987
31	1 0.4691	4 0.5368	8 0.6632	10 0.7879	10 0.8827	10 0.9421	10 0.9746	10 0.9900	10 0.9965	10 0.9989
32	1 0.4705	4 0.5403	8 0.6696	10 0.7952	10 0.8881	10 0.9456	10 0.9765	10 0.9910	10 0.9969	10 0.9991
33	1 0.4719	5 0.5436	9 0.6757	10 0.8019	11 0.8947	11 0.9506	11 0.9795	11 0.9925	11 0.9976	11 0.9993
34	1 0.4732	5 0.5473	9 0.6821	11 0.8087	11 0.8997	11 0.9537	11 0.9811	11 0.9932	11 0.9979	11 0.9994
35	1 0.4744	5 0.5508	9 0.6879	11 0.8151	11 0.9043	11 0.9564	11 0.9826	11 0.9939	11 0.9981	11 0.9995
36	1 0.4756	6 0.5542	10 0.6940	12 0.8212	12 0.9101	12 0.9605	12 0.9848	12 0.9949	12 0.9985	12 0.9996
37	1 0.4767	6 0.5578	10 0.6998	12 0.8274	12 0.9143	12 0.9629	12 0.9860	12 0.9954	12 0.9987	12 0.9997
38	1 0.4778	6 0.5613	11 0.7053	12 0.8330	12 0.9181	12 0.9651	12 0.9871	12 0.9958	12 0.9988	12 0.9997
39	1 0.4788	7 0.5647	11 0.7111	13 0.8387	13 0.9231	13 0.9683	13 0.9887	13 0.9965	13 0.9991	13 0.9998
40	1 0.4798	7 0.5683	11 0.7166	13 0.8441	13 0.9266	13 0.9703	13 0.9896	13 0.9969	13 0.9992	13 0.9998

Table A3.6-II

Optimal Design[1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 3$, $k = 3$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}, \quad b = 3f_0 + f_1$$

Confidence Coefficient (1-α)	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	492 164	123 41	55 18	31 10	21 7	15 5	11 3	9 3	7 2	6 2
0.95	296 98	75 25	33 11	19 6	12 4	9 3	7 2	6 2	4 1	3 1
0.90	214 71	54 18	24 8	14 4	9 3	6 2	6 2	4 1	3 1	3 1
0.85	165 55	42 14	19 6	11 3	7 2	6 2	4 1	3 1	3 1	3 1
0.80	131 41	33 10	15 5	9 3	6 2	4 1	3 1	3 1	3 1	3 1

[1/] The upper entry in each cell is b, and the lower entry is $\hat{f}_0$.

APPENDIX 4

Tables of approximate (continuous) optimal designs

Tables A4.1-I and A4.1-II ($p = 2(1)6$, $k = 2$)

Tables A4.2-I and A4.2-II ($p = 3$, $k = 3$)

Table A4.1-I

Continuous Optimal Designs as a Function of ξ

for $k = 2$, $p = 2(1)6$

$$D_0 = \begin{Bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & p \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 & 1 & \cdots & p-1 \\ 2 & 3 & \cdots & p \end{Bmatrix}$$

ξ	$p = 2$		$p = 3$		$p = 4$	
	$\xi_0 = 0.7979$		$\xi_0 = 1.6926$		$\xi_0 = 2.5214$	
	$\hat{\gamma}$	$\hat{g}_A$	$\hat{\gamma}$	$\hat{g}_A$	$\hat{\gamma}$	$\hat{g}_A$
1.00	0.1895	0.5121				
1.25	0.3453	0.5376				
1.50	0.4522	0.5681				
1.75	0.5297	0.6011	0.0008	0.5003		
2.00	0.5881	0.6352	0.1618	0.5168		
2.25	0.6334	0.6695	0.2599	0.5391		
2.50	0.6694	0.7032	0.3375	0.5650		
2.75	0.6983	0.7358	0.4001	0.5932	0.0941	0.5094
3.00	0.7219	0.7668	0.4514	0.6226	0.1782	0.5273
3.25	0.7414	0.7959	0.4939	0.6524	0.2475	0.5493
3.50	0.7576	0.8228	0.5296	0.6822	0.3053	0.5738
3.75	0.7711	0.8475	0.5597	0.7115	0.3541	0.5998
4.00	0.7825	0.8699	0.5854	0.7399	0.3956	0.6267
4.25	0.7921	0.8899	0.6073	0.7670	0.4312	0.6540
4.50	0.8002	0.9076	0.6262	0.7927	0.4619	0.6812
4.75	0.8071	0.9230	0.6424	0.8168	0.4886	0.7080
5.00	0.8129	0.9365	0.6565	0.8391	0.5118	0.7341
5.25	0.8179	0.9480	0.6687	0.8596	0.5321	0.7593
5.50	0.8222	0.9577	0.6794	0.8783	0.5499	0.7833
5.75	0.8258	0.9659	0.6886	0.8952	0.5656	0.8059
6.00	0.8289	0.9728	0.6967	0.9103	0.5794	0.8272
6.25	0.8315	0.9784	0.7037	0.9237	0.5916	0.8469
6.50	0.8337	0.9830	0.7099	0.9355	0.6024	0.8651
6.75	0.8356	0.9868	0.7152	0.9458	0.6119	0.8818
7.00	0.8372	0.9898	0.7199	0.9548	0.6204	0.8970
7.25	0.8385	0.9922	0.7240	0.9625	0.6279	0.9107
7.50	0.8397	0.9940	0.7276	0.9690	0.6345	0.9229
8.00	0.8415	0.9966	0.7334	0.9793	0.6456	0.9435
8.50	0.8427	0.9981	0.7378	0.9865	0.6543	0.9594
9.00	0.8436	0.9990	0.7411	0.9914	0.6611	0.9714
10.00	0.8446	0.9998	0.7454	0.9967	0.6705	0.9866
11.00	0.8450	0.9999	0.7477	0.9989	0.6761	0.9942
12.00	0.8452	1.0000	0.7489	0.9996	0.6794	0.9976
13.00	0.8453	1.0000	0.7495	0.9999	0.6813	0.9991
∞	0.8453*	1.0000	0.7500*	1.0000	0.6833*	1.0000

*Largest admissible value of γ (i.e., γ^L).

Table A4.1-I (continued)

ξ	p = 5		p = 6	
	$\xi_0 = 3.2894$		$\xi_0 = 4.0073$	
	$\hat{\gamma}$	$\hat{g}_A$	$\hat{\gamma}$	$\hat{g}_A$
3.50	0.0718	0.5075		
3.75	0.1436	0.5234		
4.00	0.2040	0.5432		
4.25	0.2555	0.5655	0.0705	0.5086
4.50	0.2997	0.5894	0.1319	0.5239
4.75	0.3380	0.6142	0.1844	0.5427
5.00	0.3714	0.6396	0.2297	0.5637
5.25	0.4006	0.6651	0.2691	0.5862
5.50	0.4262	0.6905	0.3037	0.6096
5.75	0.4489	0.7153	0.3341	0.6336
6.00	0.4690	0.7395	0.3610	0.6578
6.25	0.4868	0.7628	0.3850	0.6819
6.50	0.5026	0.7851	0.4063	0.7056
6.75	0.5167	0.8062	0.4253	0.7288
7.00	0.5293	0.8261	0.4424	0.7513
7.25	0.5405	0.8446	0.4577	0.7729
7.50	0.5506	0.8619	0.4714	0.7935
7.75	0.5596	0.8777	0.4837	0.8130
8.00	0.5676	0.8923	0.4949	0.8314
8.25	0.5748	0.9055	0.5049	0.8487
8.50	0.5813	0.9175	0.5139	0.8647
8.75	0.5870	0.9283	0.5221	0.8795
9.00	0.5922	0.9379	0.5294	0.8931
9.25	0.5968	0.9465	0.5360	0.9056
9.50	0.6010	0.9541	0.5420	0.9169
10.00	0.6080	0.9666	0.5523	0.9364
10.50	0.6135	0.9761	0.5606	0.9520
11.00	0.6179	0.9832	0.5673	0.9644
12.00	0.6242	0.9920	0.5772	0.9812
13.00	0.6279	0.9965	0.5835	0.9906
14.00	0.6302	0.9985	0.5874	0.9955
15.00	0.6315	0.9994	0.5899	0.9980
16.00	0.6322	0.9998	0.5913	0.9991
∞	0.6330*	1.0000	0.5932*	1.0000

*Largest admissible value of γ (i.e., γ^L).

Table A4.1-II

Continuous Optimal Designs[1/] to Achieve a Specified Confidence Coefficient

for $p = 2(1)6, \ k = 2$

$$\hat{D}_0 = \begin{Bmatrix} 0 & 0 & & 0 \\ & & \cdots & \\ 1 & 2 & & p \end{Bmatrix}, \quad \hat{D}_1 = \begin{Bmatrix} 1 & 1 & & p-1 \\ & & \cdots & \\ 2 & 3 & & p \end{Bmatrix}$$

Confidence Coefficient $(1-\alpha)$	p				
	2	3	4	5	6
0.99	7.0218 0.8373	8.8362 0.7401	10.3624 0.6729	11.7052 0.6226	12.9186 0.5831
0.95	5.2989 0.8188	6.8621 0.7174	8.1885 0.6491	9.3623 0.5987	10.4276 0.5595
0.90	4.3894 0.7968	5.8265 0.6912	7.0531 0.6220	8.1429 0.5718	9.1349 0.5331
0.85	3.7764 0.7724	5.1219 0.6631	6.2909 0.5935	7.3259 0.5437	8.2700 0.5056
0.80	3.2870 0.7440	4.5741 0.6312	5.6833 0.5616	6.6750 0.5126	7.5817 0.4756

[1/] The upper entry in each cell is $\hat{\xi}$, and the lower entry is $\hat{\gamma}$.

Table A4.2-I

Continuous Optimal Designs as a Function of ξ

for $p = 3$, $k = 3$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$\xi_0 = 1.4658$, $\xi_1 = 4.5081$

ξ	$\hat{\gamma}$	$\hat{g}_A$	ξ	$\hat{\gamma}$	$\hat{g}_A$	ξ	$\hat{\gamma}$	$\hat{g}_A$
1.50	0.0361	0.5008	4.25	0.9773	0.8310	7.00	1.0000	0.9802
1.75	0.2549	0.5184	4.50	0.9993	0.8554	7.25	1.0000	0.9844
2.00	0.4199	0.5450	4.75	1.0000	0.8772	7.50	1.0000	0.9879
2.25	0.5454	0.5759	5.00	1.0000	0.8965	7.75	1.0000	0.9906
2.50	0.6441	0.6092	5.25	1.0000	0.9135	8.00	1.0000	0.9928
2.75	0.7231	0.6435	5.50	1.0000	0.9283	8.50	1.0000	0.9959
3.00	0.7873	0.6780	5.75	1.0000	0.9410	9.00	1.0000	0.9977
3.25	0.8401	0.7118	6.00	1.0000	0.9518	10.00	1.0000	0.9993
3.50	0.8839	0.7444	6.25	1.0000	0.9610	11.00	1.0000	0.9998
3.75	0.9205	0.7754	6.50	1.0000	0.9686			
4.00	0.9513	0.8043	6.75	1.0000	0.9750	∞	1.0000	1.0000

Table A4.2-II

Continuous Optimal Designs[1] to Achieve a Specified Confidence Coefficient

for $p = 3$, $k = 3$

$$D_0 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{Bmatrix}, \quad D_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

Confidence Coefficient (1-α)					
0.75	0.80	0.85	0.90	0.95	0.99
3.5438	3.9613	4.4426	5.0482	5.9551	7.6870
0.8907	0.9468	0.9946	1.0000	1.0000	1.0000

[1] The upper entry in each cell is $\hat{\xi}$, and the lower entry is $\hat{\gamma}$.

REFERENCES

[1] Bechhofer, R.E. (1969). Optimal allocation of observations when comparing several treatments with a control. Multivariate Analysis II (Ed. P.R. Krishnaiah), New York.

[2] Bechhofer, R.E. and Nocturne, D.J.-M. (1972). Optimal allocation of observations when comparing several treatments with a control (II): 2-sided comparisons. Technometrics, 14, 423-436.

[3] Bechhofer, R.R. and Tamhane, A.C. (1979). Incomplete block designs for comparing treatments with a control (I): General Theory. (Submitted for publication.)

[4] Bechhofer, R.E. and Tamhane, A.C. (1979). Incomplete block designs for comparing treatments with a control (III): Optimal designs for p = 4.5, k = 3. (In preparation.)

[5] Gupta, S.S. (1963). Probability integrals of multivariate normal and multivariate t. Ann. Math. Statist., 34, 792-828.

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
#425		

4. TITLE (and Subtitle) INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL (II): OPTIMAL DESIGNS FOR $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$.	5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) ROBERT E BECHHOFER and AJIT C. TAMHANE	8. CONTRACT OR GRANT NUMBER(s) DAAG29-77-C-0003 N00014-75-C-0586 NSF ENG 77-06112
9. PERFORMING ORGANIZATION NAME AND ADDRESS School of Operations Research and Industrial Engineering, College of Engineering, Cornell University, Ithaca, N.Y. 14853.	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS National Science Foundation Washington, D.C. 20550	12. REPORT DATE May 1979 13. NUMBER OF PAGES 65
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Sponsoring Military Activities: U.S. Army Research Office, P.O. Box 12211, Research Triangle Park N.C. 27709 and Statistics and Probability Program, Office of Naval Research, Arlington, VA 22217	15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Multiple comparisons with a control, balanced treatment incomplete block (BTIB) designs, admissible designs, optimal designs, applications of equi-correlated multivariate normal and multivariate Student's t distributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we continue the study of balanced treatment incomplete block (BTIB) designs which we initiated in [3]. These designs are appropriate for comparing simultaneously $p \geq 2$ test treatments with a control treatment--the so-called multiple comparisons with a control (MCC) problem. This class of designs was characterized in [3]. In the present paper we obtain optimal designs within this class for selected (p,k,b) where $k < p+1$ is the number of plots per block, and b is the total number of blocks available. Specifically, optimal designs are obtained for $p = 2(1)6$, $k = 2$ and for

p = 3, k = 3.

Tables of exact (discrete) optimal designs are given for these (p,k)-values for a range of b-values which would ordinarily be of practical interest. Tables of approximate (continuous) optimal designs are given for situations in which very large b-values are required. The theory underlying these approximate designs is developed, and the goodness of the approximation is studied.

[3] Bechhofer, R.E. and Tamhane, A.C.: "Incomplete block design for comparing treatments with a control (I): general theory," Technical Report No. 414, School of Operations Research and Industrial Engineering, Cornell University, March 1979.